ON A CONJECTURE OF ERDŐS ABOUT ADDITIVE FUNCTIONS

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ABSTRACT. For a real-valued additive function \( f : \mathbb{N} \to \mathbb{R} \) and for each \( n \in \mathbb{N} \) we define a distribution function

\[
F_n(x) := \frac{1}{n} \# \{m \leq n : f(m) \leq x\}.
\]

In this paper we prove a conjecture of Erdős, which asserts that in order for the sequence \( F_n \) to be (weakly) convergent, it is sufficient that there exist two numbers \( a < b \) such that \( \lim_{n \to \infty} (F_n(b) - F_n(a)) \) exists and is positive.

The proof is based upon the use of the Stone–Čech compactification of \( \mathbb{N} \) to mimic the behaviour of an additive function as a sum of independent random variables.

1. INTRODUCTION

A function \( f : \mathbb{N} \to \mathbb{R} \) is called additive if \( f(mn) = f(m) + f(n) \) for any coprime integers \( m \) and \( n \). Then \( f \) is defined by its values \( f(p^k) \) on prime powers \( p^k \) (\( p \) prime, \( k \in \mathbb{N} \)) and \( f(1) = 0 \).

Given a real-valued additive function \( f \), one can define, for each \( n \in \mathbb{N} \), a distribution function

\[
F_n(x) := \frac{1}{n} \# \{m \leq n : f(m) \leq x\}. \tag{1.1}
\]

An old conjecture of Erdős in 1947 (see Erdős [4]) asserts that in order for the sequence \( F_n \) to be (weakly) convergent (in this case we say that the additive function \( f \) possesses a limit distribution), it is sufficient that there exist two numbers \( a < b \) such

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that
\[ \lim_{n \to \infty} (F_n(b) - F_n(a)) \] exists and is positive. \hfill (1.2)

In 1992 A. Hildebrand [6] could show that the conclusion of Erdős’ conjecture is valid, provided (1.2) is strengthened to
\[ L_a := \lim_{n \to \infty} F_n(a) \quad \text{and} \quad L_b := \lim_{n \to \infty} F_n(b) \] both exists and \( L_a \neq L_b \). Some further discussions are contained in [10] and [11].

In this paper we show that the above conjecture of Erdős holds.

Theorem. Let \( f : \mathbb{N} \to \mathbb{R} \) be an additive function. In order for the distributions (1.1) to converge, it is sufficient that (1.2) holds for some \( a < b \).

The proof is based upon a method, introduced in [7, 8] using the Stone–Čech compactification \( \beta \mathbb{N} \) of \( \mathbb{N} \) to mimic the behaviour of an additive function as a sum of independent random variables.

2. Finitely distributed additive functions

An additive function \( f \) is said to be finitely distributed if there are positive constants \( c_1 \) and \( c_2 \), and an unbounded sequence \( n_1 < n_2 < \ldots \) so that for every \( i \) there exists a sequence
\[ a_1^{(i)} < a_2^{(i)} < \cdots < a_t^{(i)} < n_i \]
satisfying
\[ |f(a_r^{(i)}) - f(a_s^{(i)})| < c_1, \quad t_i > c_2 n_i, \quad 1 \leq r, s \leq t_i. \]

The necessary and sufficient condition that \( f \) should be finitely distributed is that there should exist a constant \( c \) and an additive function \( h \) so that
\[ f(n) = c \log n + h(n) \] \hfill (2.1)
where both the series
\[ \sum_{|h(p)| > 1} \frac{1}{p}, \quad \sum_{|h(p)| \leq 1} \frac{h^2(p)}{p} \] converge (Erdős [3], 1946). Further characterizations of finitely distributed additive functions can be found in Ch. 7 of Elliott’s book [2]. For our purpose we shall apply the following ([2, p. 259]).

Proposition. If the additive function has a representation (2.1) with convergent series (2.2), then, if we define
\[ \alpha(n) = c \log n + \sum_{p \leq n} \frac{h(p)}{p}, \] \hfill (2.3)
the distribution functions
\[ G_\alpha(x) := \frac{1}{n} \# \{ m \leq n : f(m) - \alpha(n) \leq x \} \] \hfill (2.4)
weakly converge to some distribution function \( G(x) \).

If (1.2) holds then \( f \) is finitely distributed. Now, assume that (2.1) holds and \( \alpha(n) \) is unbounded. Then, if \( \alpha(n'_k) \to \infty, k \to \infty \), for some subsequence \( (n'_k) \), by (1.2),
\[ \lim_{k \to \infty} \left( G_{n'_k}(b - \alpha(n'_k)) - G_{n'_k}(a - \alpha(n'_k)) \right) = \lim_{k \to \infty} \left( F_{n'_k}(b) - F_{n'_k}(a) \right) > 0 \] \hfill (2.5)
whereas the left side in (2.5) tends to zero since $G_n$ converge weakly to some distribution function. Then, since $\alpha(n) = c \log n + O(\log \log n)$, we conclude $c = 0$, i.e. $f = h$, and
\[
A(n) := \sum_{p \leq n \atop |f(p)| \leq 1} \frac{f(p)}{p} = O(1) \quad \text{for all } n \in \mathbb{N}.
\] (2.6)

In the following we assume that
\[
\sum_{p \leq n \atop |f(p)| \leq 1} \frac{f(p)}{p} \text{ diverges},
\] (2.7)
which implies (see [3], Theorem II) that $G(x)$ is continuous and strictly increasing for all $x \in \mathbb{R}$.

For each $n \in \mathbb{N}$ define the additive function $f_n$ by
\[
f_n(p^k) = \begin{cases} f(p^k) & \text{if } p \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]
and put, for $A \subset \mathbb{N}$,
\[
\delta_n(A) := \frac{1}{n} \# \{ m \leq n : m \in A \}.
\]
If the limit
\[
\delta(A) := \lim_{n \to \infty} \delta_n(A)
\] (2.8)
exists we say that $A$ possesses the asymptotic density $\delta(A)$.

If some sequence $\{n'_k\}$ is given we write
\[
\delta'(A) := \lim_{k \to \infty} \delta_{n'_k}(A)
\] (2.9)
in the case the limit (2.9) exists.

With these notations we show

**Lemma 1.** Assume that (1.2) holds. Then
\[
\lim_{n \to \infty} \delta(\{m \in (a, b] : f_n(m) \in (a, b]\}) = \delta(\{m \in (a, b] : f(m) \in (a, b]\}) =: c_0 > 0.
\] (2.10)

**Proof.** Observe that $\delta(\{m \in (a, b] : f_n(m) \in (a, b]\})$ always exists. Assume that (2.10) does not hold. Then there exists a sequence $\{n_k\}$ of natural numbers such that
\[
\lim_{k \to \infty} \delta(\{m \in (a, b] : f_{n_k}(m) \in (a, b]\}) = c' \neq c_0.
\]
Since $A(n_k) = O(1)$ there exists some subsequence $\{n'_k\}$ of $\{n_k\}$ so that
\[
\lim_{k \to \infty} A(n'_k) =: \alpha'
\]
exists. Choose $k_1$ such that for every $k_0 \geq k_1$
\[
\left| \delta \left( \{m \in (a, b] : f_{n'_k}(m) \in (a, b]\} \right) - c_0 \right|
= \left| \delta' \left( \{m \in (a, b] : f_{n'_k}(m) \in (a, b]\} \right) - \delta' \left( \{m \in (a, b] : f(m) \in (a, b]\} \right) \right|
\geq \frac{|c_0 - c'|}{2}.
\] (2.11)

On the other hand we shall show that
\[
\lim_{k \to \infty} \lim_{k \to \infty} \delta_{n'_k} \left( \{m \in (a, b] : f(m) - f_{n'_k}(m) > \epsilon \} \right) = 0
\] (2.12)
for every $\epsilon > 0$ which contradicts (2.11).
For the proof of (2.12) put
\[ P_0 := \{ p: |f(p)| > 1 \} \cup \{ p^k: k \geq 2 \}. \]
Define the functions
\[ h'(m) := \sum_{\substack{p^k|m \\ p \in P_0 \\ p > n_{k_0}'}} f(p^k) \]
and
\[ j(m) := \sum_{\substack{p|m \\ |f(p)| \leq 1 \\ p > n_{k_0}'}} f(p). \]
From our definitions of these functions
\[ f(m) - f_{n_{k_0}'}(m) = j(m) - (A(n_k') - A(n_{k_0}')) + h'(m) + (A(n_k') - A(n_{k_0}')). \]
We shall prove that for every \( \varepsilon > 0 \) each of the three expressions
\[ L_1(k_0) = \lim_{k \to \infty} \delta_{n_k'}(\{ m: |j(m) - (A(n_k') - A(n_{k_0}'))| > \varepsilon \}), \]
\[ L_2(k_0) = \lim_{k \to \infty} \delta_{n_k'}(\{ m: |h'(m)| > \varepsilon \}) \]
and
\[ L_3(k_0) = \lim_{k \to \infty} \delta_{n_k'}(\{ m: |A(n_k') - A(n_{k_0}')| > \varepsilon \}) \]
converge to zero as \( k_0 \to \infty \). We may readily estimate the first of these three expressions by appealing to the Turan–Kubilius inequality. In our present circumstances it becomes
\[ \frac{1}{n_k'} \sum_{m=1}^{n_k'} |j(m) - (A(n_k') - A(n_{k_0}'))|^2 \leq \sum_{n_{k_0}' < p \leq n_k'} \frac{|f(p)|^2}{p}. \]
Appealing to the convergence of the second sum in (2.2) we see that
\[ L_1(k_0) \leq \frac{1}{\varepsilon^2} \sum_{n_{k_0}' < p \leq n_k'} \frac{|f(p)|^2}{p} = o(1) \quad \text{as} \quad k_0 \to \infty. \]
The estimate \( L_3(k_0) = o(1) \) as \( k_0 \to \infty \) is obvious.

If an integer \( m \) is counted in the expression \( L_2(k_0) \) it must satisfy one of two divisibility criteria.

First, it may be divisible by the square of a prime \( p > n_{k_0}' \). The frequency of these integers is at most
\[ \delta_{n_k'}(\{ m: p^2|m, p > n_{k_0}' \}) \leq \sum_{n_{k_0}' < p \leq n_k'} \frac{1}{p^2} = o(1) \quad \text{as} \quad k_0 \to \infty. \]
Next, it may be exactly divisible by a prime in the range \( n_{k_0}' < p \) for which \( |f(p)| > 1 \). From the hypothesis (2.2) we deduce that the frequencies of such integers is at most
\[ \sum_{n_{k_0}' < p \leq n_k'} \frac{1}{p} = o(1) \quad \text{as} \quad k_0 \to \infty \]
and thus \( L_2(k_0) = o(1) \) as \( k_0 \to \infty \). We have now shown that (2.12) holds and completed the proof of Lemma 1. \( \square \)
In the next step we identify the additive function $f$ with a sum $\sum_{p \text{ prime}} X_p$ of independent random variables.

### 3. Additive functions as a sum of independent random variables

For the sake of simplicity we restrict ourselves to strongly additive functions. Then $f$ can be written in the form

$$f = \sum_p f(p)e_p$$

where

$$e_p(n) = \begin{cases} 1 & \text{if } p|n, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ denotes the algebra generated by the sets

$$A_p := \{ n \in \mathbb{N} : p|n \}, \quad p \text{ prime},$$

then obviously each $A \in \mathcal{A}$ possesses an asymptotic density $\delta(A)$ and $\delta(A_p) = 1/p$ ($p$ prime). Thus $\delta$ defines a content on $\mathcal{A}$. Now the construction runs as follows. (For details see [7, 8].) We embed $\mathbb{N}$, endowed with the discrete topology, in the Stone–Čech compactification $\beta\mathbb{N}$, $\mathbb{N} \hookrightarrow \beta\mathbb{N}$ and, if for any $A \subset \mathbb{N}$, the closure of $A$ in $\beta\mathbb{N}$ is denoted by $\bar{A}$, then

$$\bar{A} := \{ \bar{A} \subset \beta\mathbb{N} : A \in \mathcal{A} \}$$

is an algebra, too. The extension $\bar{\delta}$ of $\delta$ by

$$\bar{\delta}(\bar{A}) := \delta(A), \quad \bar{A} \in \bar{\mathcal{A}},$$

defines a premeasure on $\bar{\mathcal{A}}$ and leads to a measure $\bar{\delta}$, induced by

$$\bar{\delta}^*(A) := \lim_{n \to \infty} \delta_n(A) \quad \text{for all } A \subset \mathbb{N},$$

and to a probability space $(\Omega, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ with $\Omega = \beta\mathbb{N}$ and with $\bar{\delta}(\bar{A}_p) = 1/p$, $p$ prime.

There is a unique extension of $e_p$ to a function $e_\bar{p}$ on $\bar{\mathcal{A}}$, and putting $X_p = f(p)e_\bar{p}$

$$f = \sum_p f(p)e_p \Rightarrow X = \sum_p f(p)e_p = \sum_p X_p$$

with

$$f_n \Rightarrow S_n := \sum_{p \leq n} X_p$$

$$P(X_p = f(p)) = \frac{1}{p}$$

and

$$P(X_p = 0) = 1 - \frac{1}{p}. $$

The $e_\bar{p}$ are independent, i.e. $X = \sum_p X_p$ is a sum of independent random variables. If (1.2) holds then, by Lemma 1,

$$\lim_{n \to \infty} P(S_n \in (a, b]) = c_0 > 0$$

and, by Proposition, $\sum_p X_p$ is essentially convergent (for the definition see [13, p. 262]).

Putting

$$a_p = \mathbb{E}(X_p^\circ), \quad Y_p = X_p - a_p, \quad T_n := \sum_{p \leq n} Y_p$$
then \( \lim_{n \to \infty} T_n \) holds a.s.. (Here \( X_p^c \) denotes the truncation of \( X_p \) at (a positive) \( c \), i.e. we replace \( X_p \) by \( X_p = X \) or 0 according as \( |X_p| < c \) or \( |X_p| \geq c \).) Denote \( Y := \lim T_n \) a.s.

It is well-known that the a.s. convergence of \( Y = \sum p Y_p \) is equivalent to the weak convergence of the distributions of the partial sums of that series. Moreover, by Kolmogorov’s three series theorem, \( Y = \sum p Y_p \) converges a.s. if and only if the series

\[
\sum_p E(Y_p^c), \quad \sum_p P(|Y_p| > c), \quad \sum_p \text{Var}(Y_p^c)
\]

(3.1)

converge.

We choose \( c = 1 \), i.e. \( a_p = E(X_p^1) \) and put (see (2.6))

\[
A(n) = \sum_{p \leq n} a_p.
\]

Then \( A(n) = O(1) \) and, the divergence of the sequence \( A(n) \) implies (see [3, Theorem 2]).

**Lemma 2.** Let \( Y = \sum p Y_p \) with \( Y_p = X_p - a_p \) as above, where the partial sums \( \sum_{p \leq N} a_p \) are bounded and divergent. Then the distribution function \( G(x) = P(Y \leq x) \) is continuous and strictly monotone for all \( x \in \mathbb{R} \).

**Remark.** The divergence of the sequence \( A(n) \) implies

\[
\sum_p a_p^+ = +\infty, \quad \sum_p a_p^- = -\infty
\]

(3.2)

where \( a_p^+ = \max(a_p, 0) \) and \( a_p^- = \max(-a_p, 0) \). Then the strict monotonicity of the distribution function \( G(x) \) in Lemma 2 can be directly proved by a result of A. Hildebrand [6].

For this we define, following the notation of Hildebrand in [6], p. 1206, the range of a random variable \( X \) as the set

\[
R(X) = \{ x \in \mathbb{R} : P(|X - x| \leq \varepsilon) > 0 \text{ for every } \varepsilon > 0 \},
\]

that is, it is equal to the set of points of increase of the distribution function \( F(x) = P(X \leq x) \). The form of this set was described by A. Hildebrand in Lemma 2 of [6] when \( X \) is given as an a.s. convergent series of independent random variables. A special version of this result is contained in the following lemma.

**Lemma 3.** Let \( \sum_{n=0}^{\infty} X_n \) be an a.s. convergent series of independent random variables and let \( X \) denote its sum. Suppose that for every \( \varepsilon > 0 \) and \( n \geq n_0 = n_0(\varepsilon) \) there exist numbers \( c_n^- = c_n^-(\varepsilon), c_n^+ = c_n^+(\varepsilon) \in R(X_n) \) with \( |c_n^-| \leq \varepsilon \) and \( |c_n^+| \leq \varepsilon \) such that

\[
\lim_{N \to \infty} \sum_{n=n_0}^{N} c_n^- = -\infty
\]

and

\[
\lim_{N \to \infty} \sum_{n=n_0}^{N} c_n^+ = +\infty.
\]

Then \( R(X) = \mathbb{R} \).

Now it is easy to prove the assertions of Lemma 2. Put

\[
c_p^- = \begin{cases} f(p) - a_p & \text{if } -\frac{f}{2} \leq f(p) < 0, \\ 0, & \text{otherwise.} \end{cases}
\]
and
\[ c^+_p = \begin{cases} f(p) - a_p & \text{if } 0 < f(p) \leq \frac{\varepsilon}{p}, \\ 0, & \text{otherwise.} \end{cases} \]

Then obviously, \( c^+_p, c^-_p \in R(Y_p), |c^-_p| \leq \frac{\varepsilon}{p} + |a_p| \leq \varepsilon \) and \( |c^+_p| \leq \frac{\varepsilon}{p} + |a_p| \leq \varepsilon \) for \( p > n_0 = n_0(\varepsilon) \) since \(|a_p| \leq 1/p\). Further,
\[
\sum_{n_0 \leq p \leq N} c^-_p = \sum_{n_0 \leq p \leq N} f(p) - \sum_{n_0 \leq p \leq N} a_p \\
< \sum_{n_0 \leq p \leq N} \frac{f(p)}{p} + O(1) \\
< \sum_{n_0 \leq p \leq N} \frac{f(p)}{p} + O(1) \\
= \sum_{n_0 \leq p \leq N} a_p + O(1) \to -\infty \quad \text{as } N \to \infty.
\]

Here the last inequality holds because of the convergence of the second series in (3.1). Similarly,
\[
\lim_{N \to \infty} \sum_{n_0 \leq p \leq N} c^+_p = +\infty.
\]

We use Lemma 3 and recall that the divergence of the series (2.7) implies, by Levy’s theorem, the continuity of \( G(x) \) to end the proof of Lemma 2.

This ends the remark.

For every subsequence \( n' = (n'_k) \) of the natural numbers we defined
\[
\delta'(A) = \lim_{k \to \infty} \delta_{n'_k}(A)
\]
if the limit exists. This leads to a content \( \delta' \) on \( \mathcal{A} \) and a measure \( P' \) on \( \beta N \) induced by
\[
\delta'^*(A) = \lim_{k \to \infty} \delta_{n'_k}(A) \quad \text{for all } A \subset \mathbb{N}.
\]

Obviously, if \( \Omega_0 \subset \beta N \) is \( P \)-measurable it is \( P' \)-measurable and \( P(\Omega_0) = P'(\Omega_0) \).

Since every bounded real-valued function \( g \) on \( \mathbb{N} \) extends uniquely to a (continuous) function \( \tilde{g} \) on \( \beta N \) (for details see R. Walker [14, p. 8 et seq.]), we conclude
\[
\Omega_0 := \{ m : f(m) \in (a, b]\} = \{ \omega : \tilde{f}(\omega) \in [a, b] \},
\]
where \( \tilde{f} \) is the unique extension of the (bounded) function \( f_{(a, b)} \), defined by
\[
f_{(a, b)}(m) = \begin{cases} f(m), & \text{if } f(m) \in (a, b], \\ |a| + |b| + 1, & \text{if } f(m) \notin (a, b]. \end{cases}
\]

If (1.2) holds then
\[
P(\Omega_0) = c_0 > 0.
\]

4. PROOF OF THE CONJECTURE OF ERDŐS

We suppose that \( A(n) \) is not convergent so that
\[
\Delta := \liminf_{n \to \infty} A(n) < \limsup_{n \to \infty} A(n) =: \overline{A}, \tag{4.1}
\]
and we shall show that this leads to a contradiction.
We fix two increasing sequences $n' = \{n'_k\}$ and $n'' = \{n''_k\}$ of positive integers so that

\[ A = \lim_{k \to \infty} A(n''_k) \quad \text{and} \quad \bar{A} = \lim_{k \to \infty} A(n'_k). \]

We put

\[ g_n = \sum_{p \leq n} (f(p)e_p - a_p) \]

and define

\[ g' = g_n' + \sum_{k=1}^{\infty} (g_{n'_k+1} - g_{n'_k}). \quad (4.2) \]

Then

\[ \{m: g'(m) \in (a - \bar{A}, b - \bar{A})\} = \{m: f(m) \in (a, b)\} \]

since $g'(m) = f(m) - \bar{A}$ for every $m \in \mathbb{N}$. Further

\[ \delta'(\{m: g'(m) \in (a - \bar{A}, b - \bar{A})\}) = \lim_{k \to \infty} \delta'(\{m: g_{n'_k}(m) \in (a - \bar{A}, b - \bar{A})\}) = c_0. \]

In the same way we define

\[ g'' = g_n'' + \sum_{k=1}^{\infty} (g_{n''_k+1} - g_{n''_k}) \]

with $g''(m) = f(m) - A$, $m \in \mathbb{N}$, and obtain

\[ \delta''(\{m: g''(m) \in (a - \bar{A}, b - \bar{A})\}) = \lim_{k \to \infty} \delta''(\{m: g_{n''_k}(m) \in (a - \bar{A}, b - \bar{A})\}) = c_0. \]

Defining the corresponding extensions $g$ and $g''$, and $P'$ and $P''$, respectively, we arrive at

\[ \Omega_0 = \{\omega: g'(\omega) \in [a - \bar{A}, b - \bar{A}]\} = \{\omega: g''(\omega) \in [a - \bar{A}, b - \bar{A}]\} \]

and

\[ P'(\{\omega: g'(\omega) \in [a - \bar{A}, b - \bar{A}]\}) = P''(\{\omega: g''(\omega) \in [a - \bar{A}, b - \bar{A}]\}) = c_0. \]

Since

\[ g' \text{ corresponds to } Y' = \lim_{k \to \infty} T_{n'_k}, \]

\[ g'' \text{ corresponds to } Y'' = \lim_{k \to \infty} T_{n''_k} \]

and since

\[ Y = \sum_{p} Y_p = \lim_{n \to \infty} T_n \]

converges a.s. with respect to $\mathbb{P}$ and possesses an everywhere continuous distribution function we conclude

(i) $\{\omega: Y'(\omega) \in [a - \bar{A}, b - \bar{A}]\} = \Omega_0$ with $P'(\Omega_0 \setminus \Omega'_0) = 0$,

(ii) $P'(\{\omega: Y'(\omega) \in [a - \bar{A}, a - \bar{A}]\}) \leq P'([\omega: Y'(\omega) \neq Y''(\omega)]) = 0$ and

(iii) $P'(\{\omega: Y'(\omega) \in [a - \bar{A}, b - \bar{A}]\}) = c_0$.

Observe, that (iii) implies that

\[ a - \bar{A} < b - \bar{A}. \]

Since $P(\{\omega: Y(\omega) \in [a - \bar{A}, a - \bar{A}]\})$ exists it must be zero by (ii), i.e.

\[ P(\{\omega: Y'(\omega) \in [a - \bar{A}, a - \bar{A}]\}) = 0. \quad (4.3) \]

In the same way we show

\[ P(\{\omega: Y(\omega) \in [b - \bar{A}, b - \bar{A}]\}) = 0. \quad (4.4) \]

(4.3) and (4.4) contradict the monotonicity of $G(x)$, and thus the assertion of Theorem 1 holds.
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References