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COUPLING AND ERGODIC THEOREMS FOR MARKOV CHAINS WITH DAMPING COMPONENT

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Abstract. Perturbed Markov chains are popular models for description of information networks. In such models, the transition matrix $\mathbf{P}_0$ of an information Markov chain is usually approximated by matrix $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$, where $\mathbf{D}$ is a so-called damping stochastic matrix with identical rows and all positive elements, while $\varepsilon \in [0, 1]$ is a damping (perturbation) parameter. Using procedure of artificial regeneration for the perturbed Markov chain $\eta_{\varepsilon,n}$, with the matrix of transition probabilities $\mathbf{P}_\varepsilon$, and coupling methods, we get ergodic theorems, in the form of asymptotic relations for $p_{\varepsilon,ij}(n) = \mathsf{P}_i\{\eta_{\varepsilon,n} = j\}$, as $n \to \infty$ and $\varepsilon \to 0$, and explicit upper bounds for the rates of convergence in such theorems. In particular, the most difficult case of the model with singular perturbations, where the phase space of the unperturbed Markov chain $\eta_{0,n}$ split in several closed classes of communicative states and possibly a class of transient states, is investigated.

Key words and phrases. Markov chain, Damping component, Information network, Regular perturbation, Singular perturbation, Coupling, Ergodic theorem, Rate of convergence, Triangular array mode.

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1. Introduction

Perturbed Markov chains is one of the popular and important objects of studies in the theory of Markov processes and their applications to stochastic networks, queuing and reliability models, bio-stochastic systems, and many other stochastic models.

We refer here to some recent books and papers devoted to perturbation problems for Markov type processes [5, 6, 11, 13, 14, 21, 24, 25, 28–30, 33–36, 38, 39, 44–46, 48–55, 57, 58]. In particular, we would like to mention works [5, 25, 48, 49], where the extended bibliographies of works in the area and the corresponding methodological and historical remarks can be found.

We are especially interested in models of Markov chains commonly used for description of information networks. With recent advancement in technology, filtering information has become a challenge in such systems. Moreover, their significance is visible as they find their applications in Internet search engines, biological, financial, transport, queuing networks and many others [1–10, 12, 15–19, 22, 27, 31, 56]. In such models an information network is represented by the Markov chain associated with the corresponding node links graph. Stationary distributions and other related characteristics of information Markov chains usually serve as basic tools for ranking of nodes in information networks.

The ranking problem may be complicated by singularity of the corresponding information Markov chain, where its phase space is split into several weakly or completely non-communicating groups of states. In such models, the matrix of transition probabilities $\mathbf{P}_0$ of information Markov chain is usually regularised and approximated by the stochastic matrix $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$, where $\mathbf{D}$ is a so-called damping stochastic matrix with identical rows and all positive elements, while $\varepsilon \in [0, 1]$ is a damping parameter.

Let $\bar{\pi}_\varepsilon$ be the stationary distribution of a Markov chain $X_{\varepsilon,n}$ with the regularised matrix of transition probabilities $\mathbf{P}_\varepsilon$. The power method is often used to approximate the corresponding stationary distribution $\bar{\pi}_0$ by rows of matrix $\mathbf{P}_\varepsilon^n$. The damping parameter $\varepsilon \in (0, 1]$ should be chosen neither too small nor too large. In the first case, where ε takes too small values, the damping effect will not work against absorbing and pseudo-absorbing effects, since the second eigenvalue for such matrices (determining the rate of convergence in the above mentioned ergodic approximation) take values approaching 1. In the second case, the ranking information (accumulated by matrix $\mathbf{P}_0$ via the corresponding stationary distribution) may be partly lost, due to the deviation of matrix $\mathbf{P}_\varepsilon$ from matrix $\mathbf{P}_0$. This actualises the problem of studies of asymptotic behaviour of stationary distributions $\bar{\pi}_\varepsilon$ as $\varepsilon \to 0$ and matrices $\mathbf{P}_\varepsilon^n$ in triangular array mode, where $\varepsilon \to 0$ and $n \to \infty$, simultaneously.

The model, where matrix $\mathbf{P}_0$ is a matrix of transition probabilities for a Markov chain, which phase space is one class of communicative states is usually referred as the model with regular perturbations. The model, where matrix $\mathbf{P}_0$ is a matrix of transition probabilities for a Markov chain, which phase space split in several closed classes of communicative states plus a class (possibly empty) of transient states, is usually referred as the model with singular perturbations.

The asymptotic analysis in the singular case is much more difficult than in the regular case. The approach used in this paper based on the use of method of artificial regeneration and renewal techniques for deriving special series representation for stationary distributions $\bar{\pi}_\varepsilon$ and coupling method.

The paper includes six sections. In Section 2, we describe the algorithm for stochastic modelling of Markov chains with damping component and the procedure of embedding such Markov chains in the model of discrete time regenerative processes with special damping regenerative times. Also, we derive renewal type equations for the corresponding transition probabilities, present ergodic theorems for the Markov chains with damping component and derive special series representation for the corresponding stationary distributions. In Section 3, explicit upper bounds in approximations of the stationary distributions for Markov chain with damping component are given. In Section 4, we give coupling explicit estimate for the rate of convergence in ergodic theorems for Markov chains with damping component. In Section 5, we present ergodic theorems for Markov chains with damping component in triangular array mode. In Section 6, some concluding comments are given.

2. Markov chains with damping component (MCDC)

Let (a) $\mathbb{X} = \{1, 2, \ldots, m\}$ be a finite set, (b) $\bar{p} = \langle p_1, \ldots, p_m \rangle$ and $\bar{d} = \langle d_1, \ldots, d_m \rangle$, be two discrete probability distributions, (c) $\mathbf{P}_0 = \| p_{0,ij} \|$ be a $m \times m$ stochastic matrix, (d) $\mathbf{D} = \| d_{ij} \|$ be a $m \times m$ damping stochastic matrix with elements $d_{ij} = d_j > 0$, $i, j = 1, \ldots, m$, and (e) $\mathbf{P}_\varepsilon = \| p_{\varepsilon,ij} \| = (1 - \varepsilon)\mathbf{P}_0 + \varepsilon \mathbf{D}$ is a stochastic matrix with elements $p_{\varepsilon,ij} = (1 - \varepsilon)p_{0,ij} + \varepsilon d_j, i, j = 1, \ldots, m$, where $\varepsilon \in [0, 1]$.

We refer to a Markov chain $X_{\varepsilon,n}, n = 0, 1, \ldots$, with the phase space $\mathbb{X}$, the initial distribution $\bar{p}$, and the matrix of transition probabilities $\mathbf{P}_\varepsilon$ as a Markov chain with damping component (MCDC).

Denote by L_m the class of all initial distributions $\bar{p} = \langle p_1, \ldots, p_m \rangle$.

Let $p_{\varepsilon,ij}(n) = \mathsf{P}\{X_{\varepsilon,n} = j / X_{\varepsilon,0} = i\}, i, j \in \mathbb{X}$, $n = 0, 1, \ldots$, be n-steps transition probabilities for the Markov chain $X_{\varepsilon,n}$. Obviously, $p_{\varepsilon,ij}(0) = \mathrm{I}(i = j), i, j \in \mathbb{X}$ and $p_{\varepsilon,ij}(1) = p_{\varepsilon,ij}, i, j \in \mathbb{X}$. Let, also, $p_{\varepsilon,\bar{p},j}(n) = \mathsf{P}_{\bar{p}}\{X_{\varepsilon,n} = j\} = \sum_{i \in \mathbb{X}} p_i p_{\varepsilon,ij}(n)$, $\bar{p} \in L_m$, $j \in \mathbb{X}$, $n = 0, 1, \ldots$ Obviously, $p_{\varepsilon,\bar{p},j}(0) = p_j, j \in \mathbb{X}$.

Here and henceforth, symbols $\mathsf{P}_{\bar{p}}$ and $\mathsf{E}_{\bar{p}}$ are used for probabilities and expectations related to a Markov chain with an initial distribution $\bar{p}$. In the case, where the initial distribution is concentrated in a state i the above symbols take the forms P_i and E_i.

The phase space $\mathbb{X}$ of the Markov chain $X_{\varepsilon,n}$ is one aperiodic class of communicative states, for every $\varepsilon \in (0,1]$.

Let us describe a procedure of embedding the Markov chain $X_{\varepsilon,n}$ in the model of discrete time regenerative processes with special damping regenerative times.

Let us define random variables $U, W, U_{i,n}, V_{\varepsilon,n}, W_{\varepsilon,n}$ satisfying the following assumptions:

(a) U takes values in space $\mathbb{X}$ and $\mathsf{P}\{U = j\} = p_j, j \in \mathbb{X}$;

(b) $U_{i,n}, i \in \mathbb{X}, n = 1, 2, \ldots$, is family of independent random variables taking values in space $\mathbb{X}$ and such that $\mathsf{P}\{U_{i,n} = j\} = p_{0,ij}, i, j \in \mathbb{X}, n = 1, 2, \ldots$;

(c) $V_n, n = 0, 1, \ldots$, is a sequence of independent random variables taking values in space $\mathbb{X}$ and such that $\mathsf{P}\{V_n = j\} = d_j, j \in \mathbb{X}, n = 1, 2, \ldots$;

(d) W is a binary random variable taking values 0 and 1 with probabilities, respectively q_0 and q_1;

(e) $W_{\varepsilon,n}, n = 1, 2, \ldots$, be, for every $\varepsilon \in [0,1]$, a sequence of independent binary random variables taking values 0 and 1 with probabilities, respectively, $1-\varepsilon$ and ε, for $n = 1, 2, \ldots$;

(f) the random variables U, W, the family of random variables $U_{i,n}, i \in \mathbb{X}, n = 1, 2, \ldots$, and the random sequences $V_{\varepsilon,n}, n = 1, 2, \ldots$ and $W_{\varepsilon,n}, n = 1, , 2, \ldots$, are mutually independent, for every $\varepsilon \in [0,1]$.

Let us now define, for every $\varepsilon \in [0,1]$, the random sequence $X_{\varepsilon,n}, n = 0, 1, \ldots$, by the following recurrent relation,

$$X_{\varepsilon,n} = U_{X_{\varepsilon,n-1},n}\mathrm{I}(W_{\varepsilon,n} = 0) + V_n\mathrm{I}(W_{\varepsilon,n} = 1), n = 1, 2, \ldots, \ X_{\varepsilon,0} = U. \tag{1}$$

It is readily seen that the random sequence $X_{\varepsilon,n}, n = 0, 1, \ldots$ is, for every $\varepsilon \in [0,1]$, a homogeneous Markov chain with phase space $\mathbb{X}$, the initial distribution $\bar{p}$ and the matrix of transition probabilities $\mathbf{P}_\varepsilon$.

Let us now consider the extended random sequence,

$$Y_{\varepsilon,n} = (X_{\varepsilon,n}, W_{\varepsilon,n}), n = 1, 2, \ldots, \ X_{\varepsilon,0} = U, \ W_{\varepsilon,0} = W. \tag{2}$$

This random sequence also is, for every $\varepsilon \in [0,1]$, a homogeneous Markov chain, with phase space $\mathbb{Y} = \mathbb{X} \times \{0,1\}$, the initial distribution $\overline{p}_\circ = \langle p_i q_r, \, (i,r) \in \mathbb{Y}\rangle$ and transition probabilities,

$$p_{\varepsilon,ir,jk} = \mathsf{P}\{X_{\varepsilon,1} = j, W_{\varepsilon,1} = k / X_{\varepsilon,0} = i, W_{\varepsilon,0} = r\} =$$

$$= \begin{cases} (1-\varepsilon)p_{0,ij} & \text{for } (i,r) \in \mathbb{Y}, \, j \in \mathbb{X}, \, k = 0, \\ \varepsilon d_j & \text{for } (i,r) \in \mathbb{Y}, \, j \in \mathbb{X}, \, k = 1. \end{cases} \tag{3}$$

Now, let us assume that $\varepsilon \in (0,1]$.

Let us define times of sequential hitting state 1 by the second component $W_{\varepsilon,n}$,

$$T_{\varepsilon,n} = \min(n > T_{\varepsilon,n-1}, W_{\varepsilon,n} = 1), n = 1, 2, \ldots, \ T_{\varepsilon,0} = 0. \tag{4}$$

The random sequence $Y_{\varepsilon,n}, n = 0, 1, \ldots$, is a discrete time regenerative process with "damping" regeneration times $T_{\varepsilon,n}, n = 0, 1, \ldots$.

This follows from independence of transition probabilities $p_{\varepsilon,ir,jk}$, given by relation (3), on $(i,r) \in \mathbb{Y}$ if $k = 1$.

This is a standard regenerative process, if the initial distribution $\overline{d}_\circ = \langle d_i \mathrm{I}(r = 1), (i,r) \in \mathbb{Y}\rangle$.

The inter-regeneration times $S_{\varepsilon,n} = T_{\varepsilon,n} - T_{\varepsilon,n-1}, n = 1, 2, \ldots$, are i. i. d. geometrically distributed random variables, with parameter ε, i. e., $\mathsf{P}\{S_{\varepsilon,1} = n\} = \varepsilon(1-\varepsilon)^{n-1}, n \geq 1$.

Let us introduce sets $\mathbb{Z}_j = \{(j,0),(j,1)\}, \ j \in \mathbb{X}\}$. Obviously, $p_{\varepsilon,\bar{d},j}(n) = \mathsf{P}_{\overline{d}_\circ}\{Y_{\varepsilon,n} \in \mathbb{Z}_j\}, \ n \geq 0$. That is why, probabilities $p_{\varepsilon,\bar{d},j}(n), n \geq 0$ are, for every

$j \in \mathbb{X}$, the unique bounded solution for the following discrete time renewal equation,

$$p_{\varepsilon,\bar{d},j}(n) = p_{0,\bar{d},j}(n)(1-\varepsilon)^n + \sum_{l=1}^{n} p_{\varepsilon,\bar{d},j}(n-l)\varepsilon(1-\varepsilon)^{l-1},\ n \geq 0. \tag{5}$$

If the initial distribution $\overline{p}_\circ \neq \overline{d}_\circ$, $Y_{\varepsilon,n}$ is a regenerative process with the transition period $[0, T_{\varepsilon,1})$.

In the case, probabilities $p_{\varepsilon,\bar{p},j}(n)$ and $p_{\varepsilon,\bar{d},j}(n)$ are, for every $j \in \mathbb{X}$, connected by the following renewal type relation,

$$p_{\varepsilon,\bar{p},j}(n) = p_{0,\bar{p},j}(n)(1-\varepsilon)^n + \sum_{l=1}^{n} p_{\varepsilon,\bar{d},j}(n-l)\varepsilon(1-\varepsilon)^{l-1},\ n \geq 0. \tag{6}$$

The following theorem give a very useful series representation for the stationary distribution of the MCDC $X_{\varepsilon,n}$.

Theorem 1. *The following ergodic relation takes place for any initial distribution* $\bar{p} \in L_m$, $j \in \mathbb{X}$, *and* $\varepsilon \in (0,1]$,

$$p_{\varepsilon,\bar{p},j}(n) \to \pi_{\varepsilon,j} = \varepsilon \sum_{l=0}^{\infty} p_{0,\bar{d},j}(l)(1-\varepsilon)^l \text{ as } n \to \infty. \tag{7}$$

Proof. In the standard regeneration case, the geometrical distribution of the regeneration time $T_{\varepsilon,1} = S_{\varepsilon,1}$ is aperiodic and has expectation ε^{-1}.

This makes it possible to apply the discrete time renewal theorem (see, for example, [20]) to the renewal equation (5). This yields the following ergodic relation, for $j \in \mathbb{X}$,

$$p_{\varepsilon,\overline{d},j}(n) \to \pi_{\varepsilon,j} = \varepsilon \sum_{l=0}^{\infty} p_{0,\bar{d},j}(l)(1-\varepsilon)^l \text{ as } n \to \infty. \tag{8}$$

Obviously $p_{0,\bar{p},j}(n)(1-\varepsilon)^n \to 0$ as $n \to \infty$, for $j \in \mathbb{X}$. Let us also define $p_{\varepsilon,\overline{d},j}(n-l) = 0$, for $l > n$. Relation (8) implies that $p_{\varepsilon,\overline{d},j}(n-l) \to \pi_{\varepsilon,j}$ as $n \to \infty$, for $l \geq 0$ and $j \in \mathbb{X}$. Using the latter two asymptotic relations, relation (6), and the Lebesgue theorem, we get, for $\overline{p} \in L_m, j \in \mathbb{X}$,

$$\lim_{n\to\infty} p_{\varepsilon,\overline{p},j}(n) = \lim_{n\to\infty} p_{0,\bar{p},j}(n)(1-\varepsilon)^n + \lim_{n\to\infty} \sum_{l=1}^{\infty} p_{\varepsilon,\overline{d},j}(n-l)\varepsilon(1-\varepsilon)^{l-1} = \pi_{\varepsilon,j}. \tag{9}$$

The proof is complete. □

The phase space $\mathbb{X}$ is one aperiodic class of communicative states for the Markov chain $X_{\varepsilon,n}$, for every $\varepsilon \in (0,1]$. In this case, its stationary distribution $\bar{\pi}_\varepsilon = \langle \pi_{\varepsilon,j}, j \in \mathbb{X}\rangle$ is the unique positive solution for the system of linear equations,

$$\sum_{i\in\mathbb{X}} \pi_{\varepsilon,i} p_{\varepsilon,ij} = \pi_{\varepsilon,j}, j \in \mathbb{X},\ \sum_{j\in\mathbb{X}} \pi_{\varepsilon,j} = 1. \tag{10}$$

Also, the stationary probabilities $\pi_{\varepsilon,j}$ can be represented in the form $\pi_{\varepsilon,j} = e_{\varepsilon,j}^{-1}$, $j \in \mathbb{X}$, via the expected return times $e_{\varepsilon,j}$, with the use of regeneration property of the Markov chain $X_{\varepsilon,n}$ at moments of return in state j.

The series representation (7) for the stationary distribution of Markov chain $X_{\varepsilon,n}$ is based on the use of alternative damping regeneration times. This representation is, by our opinion, a more effective tool for performing asymptotic perturbation analysis for MCDC.

Let us also describe ergodic properties of the Markov chain $X_{0,n}$.

Its ergodic properties are determined by communicative properties its phase space $\mathbb{X}$ and the matrix of transition probabilities $\mathbf{P}_0$. The simplest case is where the following condition holds:

A$_1$: The phase space $\mathbb{X}$ is one aperiodic class of communicative states for the Markov chain $X_{0,n}$.

In this case, the following ergodic relation holds, for any $\bar{p} \in L_m, j \in \mathbb{X}$,

$$p_{0,\bar{p},j}(n) \to \pi_{0,j} \text{ as } \varepsilon \to 0, \tag{11}$$

The stationary distribution $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X} \rangle$ is the unique positive solution of the system of linear equations,

$$\sum_{i \in \mathbb{X}} \pi_{0,i} p_{0,ij} = \pi_{0,j}, j \in \mathbb{X}, \sum_{j \in \mathbb{X}} \pi_{0,j} = 1. \tag{12}$$

A more complex is the case, where the following condition holds:

B$_1$: The phase space $\mathbb{X} = \cup_{g=0}^{h} \mathbb{X}^{(g)}$, where: **(a)** $X^{(g)}, g = 0, \ldots, h$, are non-intersecting subsets of $\mathbb{X}$, **(b)** $X^{(g)}, g = 1, \ldots, h$, are non-empty, closed, aperiodic classes of communicative states for the Markov chain $X_{0,n}$, **(c)** and $X^{(0)}$ is a class (possibly empty) of transient states for the Markov chain $X_{0,n}$.

If the initial distribution of there Markov chain $X_{0,n}$ is concentrated at the set $\mathbb{X}^{(g)}$, for some $g = 1, \ldots, h$, then $X_{0,n} = X_{0,n}^{(g)}, n = 0, 1, \ldots$, can be considered as a Markov chain with the reduced phase space $\mathbb{X}^{(g)}$ and the matrix of transition probabilities $\mathbf{P}_{0,g} = \| p_{0,rk} \|_{k,r \in \mathbb{X}^{(g)}}$.

According condition **B**$_1$, there exists, for any $r, k \in \mathbb{X}^{(g)}, g = 1, \ldots, h$,

$$p_{0,rk}(n) \to \pi_{0,k}^{(g)} \text{ as } n \to \infty, \tag{13}$$

where $\bar{\pi}_0^{(g)} = \langle \pi_{0,k}^{(g)}, k \in \mathbb{X}^{(g)} \rangle$ is, for $g = 1, \ldots, h$, the stationary distribution of the Markov chain $X_{0,n}^{(g)}$.

The stationary distribution $\bar{\pi}_0^{(g)}$ is, for every $g = 1, \ldots, h$, the unique positive solution for the system of linear equations,

$$\pi_{0,k}^{(g)} = \sum_{r \in \mathbb{X}^{(g)}} \pi_{0,r}^{(g)} p_{0,rk}, k \in \mathbb{X}^{(g)}, \sum_{k \in \mathbb{X}^{(g)}} \pi_{0,k}^{(g)} = 1. \tag{14}$$

Let $Z_\varepsilon = \min(n \geq 0 : X_{\varepsilon,n} \in \overline{\mathbb{X}}^{(0)})$ be the first hitting time of the Markov chain $X_{\varepsilon,n}$ into the set $\overline{\mathbb{X}}^{(0)}$. Note that $Z_\varepsilon = 0$, if $X_{\varepsilon,0} \in \overline{\mathbb{X}}^{(0)}$, while $Z_\varepsilon \geq 1$, if $X_{\varepsilon,0} \in \mathbb{X}^{(0)}$.

Let also introduce probabilities, for $i \in \mathbb{X}$ and $g = 1, \ldots, h$,

$$f_{\varepsilon,i}^{(g)} = \mathsf{P}_i\{X_{\varepsilon,Z_\varepsilon} \in \mathbb{X}^{(g)}\}. \tag{15}$$

The following relation takes place, for $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \ldots, h$,

$$\begin{aligned} f_{\varepsilon,\bar{p}}^{(g)} = \mathsf{P}_{\bar{p}}\{X_{\varepsilon,Z_\varepsilon} \in \mathbb{X}^{(g)}\} &= \sum_{i \in \mathbb{X}^{(g)}} p_i + \sum_{i \in \mathbb{X}^{(0)}} p_i f_{\varepsilon,i}^{(g)} = \\ &= \sum_{i \in \mathbb{X}^{(g)}} p_i + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{\infty} \sum_{r \in \mathbb{X}^{(g)}} \mathsf{P}_i\{Z_\varepsilon = l, X_{\varepsilon,l} = r\}. \end{aligned} \tag{16}$$

Note that in the case, where the set $\mathbb{X}^{(0)}$ is empty, the second sum disappears in the above formulas for probabilities $f_{\varepsilon,\bar{p}}^{(g)}$.

Lemma 1. *Let condition* **B**$_1$ *holds. Then, the following ergodic relation takes place, for* $\bar{p} \in L_m$ *and* $k \in \mathbb{X}$,

$$\lim_{n \to \infty} p_{0,\bar{p},k}(n) = \pi_{0,\bar{p},k} = \begin{cases} f_{0,\bar{p}}^{(g)} \pi_{0,k}^{(g)} & \text{for } k \in \mathbb{X}^{(g)}, g = 1, \ldots, h, \\ 0 & \text{for } k \in \mathbb{X}^{(0)}. \end{cases} \tag{17}$$

Proof. Let us assume that $\mathbb{X}^{(0)}$ is a non-empty set.

The following relation takes place, for $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \dots, h$,

$$p_{0,\bar{p},k}(n) = \sum_{i\in\mathbb{X}^{(g)}} p_i p_{0,ik}(n) + \sum_{i\in\mathbb{X}^{(0)}} p_i \sum_{l=1}^{n} \sum_{r\in\mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} p_{0,rk}(n-l),\ n \geq 0. \tag{18}$$

Let us also define $p_{0,rk}(n-l) = 0$, for $l > n$. Relation (13) implies that $p_{0,rk}(n-l) \to \pi_{0,k}^{(g)}$ as $n \to \infty$, for $l \geq 0$ and $r, k \in \mathbb{X}^{(g)}, g = 1, \dots, h$.

Using the above relation, relations (16), (18) and the Lebesgue theorem, we get, for $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \dots, h$,

$$\begin{aligned}\lim_{n\to\infty} p_{0,\bar{p},k}(n) &= \lim_{n\to\infty} \sum_{i\in\mathbb{X}^{(g)}} p_i p_{0,ik}(n) + \\ &\quad + \lim_{n\to\infty} \sum_{i\in\mathbb{X}^{(0)}} p_i \sum_{l=1}^{\infty} \sum_{r\in\mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} p_{0,rk}(n-l) = \\ &= \sum_{i\in\mathbb{X}^{(g)}} p_i \pi_{0,k}^{(g)} + \sum_{i\in\mathbb{X}^{(0)}} p_i \sum_{l=1}^{\infty} \sum_{r\in\mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} \pi_{0,k}^{(g)} = \\ &= f_{0,\bar{p}}^{(g)} \pi_{0,k}^{(g)}.\end{aligned} \tag{19}$$

Also, the following relation holds, for $\bar{p} \in L_m, k \in \mathbb{X}^{(0)}$,

$$\pi_{0,\bar{p},k}(n) = \sum_{i\in\mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 > n, X_{0,n} = k\} \leq \sum_{i\in\mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 > n\} \to 0 \text{ as } n \to \infty. \tag{20}$$

The case, where $\mathbb{X}^{(0)} = \emptyset$, is trivial. □

Ergodic relation (17) shows that in the case, where condition $\mathbf{B}_1$ holds, the stationary probabilities $\pi_{0,\bar{p},k}$ defined by the asymptotic relation (17) may depend on the initial distribution.

The perturbation model, where condition $\mathbf{A}_1$ holds, i.e., the phase space $\mathbb{X}$ is one class of communicative states for the Markov chain $X_{0,n}$, can be referred as regular. The perturbation model, where condition $\mathbf{B}_1$ holds, i.e., the phase space $\mathbb{X}$ is not one class of communicative states for the Markov chain $X_{0,n}$, can be referred as singular.

3. Rate of Convergence for Stationary Distributions of Perturbed MCDC

In this section, we obtain explicit upper bounds for deviations of stationary distributions for Markov chains $X_{\varepsilon,n}$ and $X_{0,n}$.

It is well known that, under condition $\mathbf{A}_1$, the rate of convergence in the ergodic relation (17) is exponential. This means that there exist some constants $C = C(\mathbf{P}_0) \in [0, \infty)$, $\lambda = \lambda(\mathbf{P}_0) \in [0, 1)$, and distribution $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X}\rangle$, with all positive component such that the following relation holds,

$$\max_{i,j\in\mathbb{X}} |p_{0,ij}(n) - \pi_{0,j}| \leq C\lambda^n,\ n \geq 1. \tag{21}$$

In fact, condition $\mathbf{A}_1$ is equivalent to the following condition:

$\mathbf{A}_2$: There exist a constants $C = C(\mathbf{P}_0) \in [0, \infty)$, $\lambda = \lambda(\mathbf{P}_0) \in [0, 1)$, and a distribution $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X}\rangle$ with all positive component such that relation (21) holds.

Indeed, condition $\mathbf{A}_2$ implies that probabilities $p_{0,ij}(n) > 0, i, j \in \mathbb{X}$ for all large enough n. This implies that $\mathbb{X}$ is one aperiodic class of communicative states. Also, condition $\mathbf{A}_2$ implies that $p_{0,ij}(n) \to \pi_{0,j}$ as $n \to \infty$, for $i, j \in \mathbb{X}$, and, thus, $\bar{\pi}_0$ is the stationary distribution for the Markov chain $X_{0,n}$.

According the Perron–Frobenius theorem, the role of λ can play the absolute value of the second (by absolute value), eigenvalue for matrix $\mathbf{P}_0$. As far as constant C is concerned, we refer to the book [20], where one can find the algorithms which let one compute this constant.

The following theorem present explicit upper bounds for deviations of stationary distributions of Markov chains $X_{\varepsilon,n}$ and $X_{0,n}$, for the regular perturbation model.

Theorem 2. *Let condition $\mathbf{A}_2$ holds. Then the following relation holds, for $j \in \mathbb{X}$,*

$$|\pi_{\varepsilon,j} - \pi_{0,j}| \leq \varepsilon\left(|d_j - \pi_{0,j}| + \frac{C\lambda}{1-\lambda}\right), \tag{22}$$

where the damping distribution $\bar{d} = \langle d_j, j \in \mathbb{X}\rangle$ has been defined in Section 2.

Proof. The inequalities appearing in condition $\mathbf{A}_2$ imply that the following relation holds, for $n \geq 1, j \in \mathbb{X}$,

$$|p_{0,\bar{d},j}(n) - \pi_{0,j}| = |\sum_{i\in\mathbb{X}}(d_i p_{0,ij}(n) - d_i\pi_{0,j})| \leq \sum_{i\in\mathbb{X}} d_i|p_{0,ij}(n) - \pi_{0,j})| \leq C\lambda^n. \tag{23}$$

Using relations (7) and (23), we get the following estimate, for $j \in \mathbb{X}$,

$$\begin{aligned}
|\pi_{\varepsilon,j} - \pi_{0,j}| &\leq |\varepsilon\sum_{l=0}^{\infty} p_{0,\bar{d},j}(l)(1-\varepsilon)^l - \pi_{0,j}| = \\
&= |\varepsilon\sum_{l=0}^{\infty} p_{0,\bar{d},j}(l)(1-\varepsilon)^l - \varepsilon\sum_{l=0}^{\infty}\pi_{0,j}(1-\varepsilon)^l| \leq \\
&\leq \varepsilon|d_j - \pi_{0,j}| + \varepsilon\sum_{l=1}^{\infty} C\lambda^l(1-\varepsilon)^l \leq \\
&\leq \varepsilon\left(|d_j - \pi_{0,j}| + \frac{C\lambda(1-\varepsilon)}{1-\lambda(1-\varepsilon)}\right) \leq \varepsilon\left(|d_j - \pi_{0,j}| + \frac{C\lambda}{1-\lambda}\right).
\end{aligned} \tag{24}$$

The proof is complete. □

The quantities $|d_j - \pi_{0,j}|$ appearing in inequality (22) are, in some sense, determined by a prior information about the stationary probabilities. They takes smaller values if one can choose initial distribution $\bar{p}$ with smaller deviation of the stationary distribution $\bar{\pi}_0$. Inequalities $|d_j - \pi_{0,j}| \leq d_j \vee (1 - d_j) \leq 1$ let one replace the term $|d_j - \pi_{0,j}|$ in inequality (22) by quantities independent on the corresponding stationary probabilities $\pi_{0,j}$.

Theorem 2 remains also valid if condition $\mathbf{A}_2$ is weaken by omitting in it the assumption of positivity for the distribution $\bar{\pi}_0 = \langle\pi_{0,i}, i \in \mathbb{X}\rangle$ appearing in this condition. In this case, condition $\mathbf{A}_2$ implies that the phase space $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_0$, where $\mathbb{X}_1 = \{i \in \mathbb{X} : \pi_{0,i} > 0\}$ is a non-empty closed class of communicative states, while $\mathbb{X}_0 = \{i \in \mathbb{X} : \pi_{0,i} = 0\}$ is a the class (possibly empty) of transient states, for the Markov chain $X_{0,n}$. Note that $\bar{\pi}_0$ still is the stationary distribution for this Markov chain.

We would like also to refer to paper [33], where one can find alternative upper bounds for the rate of convergence of stationary distributions for perturbed Markov chains and further related references.

Let now assume that condition $\mathbf{B}_1$ holds.

Let us consider matrices, for $g = 0, \ldots, h$ and $n = 0, 1, \ldots,$

$$\mathbf{P}_{0,g} = \|p_{0,rk}\|_{r,k\in\mathbb{X}^{(g)}} \text{ and } \mathbf{P}_{0,g}^{n} = \|p_{0,rk}^{(g)}(n)\|_{r,k\in\mathbb{X}^{(g)}}. \tag{25}$$

Note that, for $g = 1,\dots,h$, probabilities $p_{0,rk}^{(g)}(n) = p_{0,rk}(n), r,k \in \mathbb{X}^{(g)}, n \geq 0$, since $\mathbb{X}^{(g)}, j = 1,\dots,h$ are closed classes of states.

The reduced Markov chain $X_{0,n}^{(g)}$ with the phase space $\mathbb{X}^{(g)}$ and the matrix of transition probabilities $\mathbf{P}_{0,g}$ is, for every $g = 1,\dots,h$, exponentially ergodic and the following estimates take place, for $k \in \mathbb{X}^{(g)}, g = 1,\dots,h$ and $n = 0,1,\dots,$

$$\max_{r,k\in\mathbb{X}^{(g)}} |p_{0,rk}^{(g)}(n) - \pi_{0,k}^{(g)}| \leq C_g\lambda_g^n, \tag{26}$$

with some constants $C_g = C_g(\mathbf{P}_0) \in [0,\infty), \lambda_g = \lambda_g(\mathbf{P}_0) \in [0,1), g = 1,\dots,h$ and distributions $\bar{\pi}_0^{(g)} = \langle \pi_{0,k}^{(g)}, k \in \mathbb{X}^{(g)}\rangle, g = 1,\dots,h$, with all positive component.

Obviously, inequalities (26) imply that $p_{0,rk}^{(g)}(n) \to \pi_{0,k}^{(g)}$ as $n \to \infty$, for $r,k \in \mathbb{X}^{(g)}$, $g = 1,\dots,h$. Thus, distribution $\bar{\pi}_0^{(g)}$ is the stationary distribution for the Markov chain $X_{0,n}^{(g)}$, for every $g = 1,\dots,h$.

As has been mentioned above the role of λ_g can play, for every $g = 1,\dots,h$, the absolute value of the second (by absolute value), eigenvalue for matrix $\mathbf{P}_{0,g}$, and C_g is the constant, which as has been mentioned above can be computed using the algorithm described in book [20].

As well known, there exists $\lambda_0 = \lambda_0(\mathbf{P}_0) \in (0,1)$ such that there exist finite exponential moments, for $i \in \mathbb{X}^{(0)}$,

$$C_{0,i} = C_{0,i}(\mathbf{P}_0) = \mathsf{E}_i e^{(\ln\lambda_0^{-1})Z_0} = \mathsf{E}_i\lambda_0^{-Z_0} < \infty. \tag{27}$$

Let us also denote,

$$C_0 = \max_{i\in\mathbb{X}^{(0)}} C_{0,i}. \tag{28}$$

The upper estimates for λ_0 can be found, for example, in book [25].

Let us denote,

$$\bar{\lambda} = \max_{0\leq g\leq h} \lambda_g, \ \bar{C} = \max_{1\leq g\leq h} (C_g + C_gC_0 + C_0). \tag{29}$$

Here, one should formally count $C_0, \lambda_0 = 0$, if the class $\mathbb{X}^{(0)}$ is empty.

Condition $\mathbf{B}_1$ is, in fact, equivalent to the following condition:

$\mathbf{B}_2$: The phase space $\mathbb{X} = \cup_{g=0}^{h}\mathbb{X}^{(g)}$, where: **(a)** $X^{(g)}, g = 0,\dots,h$, are non-intersecting subsets of $\mathbb{X}$, **(b)** $X^{(g)}, g = 1,\dots,h$, are non-empty, closed classes of states for the Markov chain $X_{0,n}$ such that inequalities (26) hold, **(c)** $X^{(0)}$ is a class of states for the Markov chain $X_{0,n}$ such that relation (27) holds (if $X^{(0)}$ is a non-empty set).

Indeed, condition $\mathbf{B}_2$ implies that probabilities $p_{0,rk}^{(g)}(n) > 0, r,k \in \mathbb{X}^{(g)}, g = 1,\dots,h$, for all large enough n. This implies that $\mathbb{X}^{(g)}, g = 1,\dots,h$ are closed aperiodic classes of communicative states. Also, inequalities (26) imply that $p_{0,rk}^{(g)}(n) \to \pi_{0,k}^{(g)}$ as $n \to \infty$, for $r,k \in \mathbb{X}^{(g)}, g = 1,\dots,h$, and, thus, $\bar{\pi}_0^{(g)} = \langle \pi_{0,k}^{(g)}, k \in \mathbb{X}^{(g)}\rangle$ is the stationary distribution for the Markov chain $X_{0,n}^{(g)}$, for every $g = 1,\dots,h$. Also, relation (27) implies that probabilities $p_{0,rk}^{(0)}(n) \to 0$ as $n \to \infty$, for $r,k \in \mathbb{X}^{(0)}$ (if $X^{(0)}$ is a non-empty set). This implies that $\mathbb{X}^{(0)}$ is a class of transient states for the Markov chain $X_{0,n}^{(0)}$.

Lemma 2. *Let condition $\mathbf{B}_2$ holds. Then the following relation holds, for $\bar{p} \in L_m, k \in \mathbb{X}$ and $n \geq 1$,*

$$|p_{0,\bar{p},k}(n) - \pi_{0,\bar{p},k}| \leq \bar{C}\bar{\lambda}^n. \tag{30}$$

Proof. Let us, first, assume that $X^{(0)} = \emptyset$.

In this case, relations (16)–(18) imply that, for $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \ldots, h$ and $n \geq 1$,

$$|p_{0,\bar{p},k}(n) - \pi_{0,\bar{p},k}| = |\sum_{i \in X^{(g)}} p_i p^{(g)}_{0,ik}(n) - \sum_{i \in X^{(g)}} p_i \pi^{(g)}_{0,k}| \leq C_g \lambda_g^n \leq \bar{C}\bar{\lambda}^n. \tag{31}$$

Let us now assume that $X^{(0)} \neq \emptyset$.

Using relations (16)–(18) and (26)–(27), we get the following inequalities, for $\bar{p} \in L_m$, $k \in \mathbb{X}^{(g)}$, $g = 1, \ldots, h$ and $n \geq 1$,

$$\begin{aligned}
|p_{0,\bar{p},k}(n) - \pi_{0,\bar{p},k}| &= |\sum_{i \in \mathbb{X}^{(g)}} p_i p^{(g)}_{0,ik}(n) + \\
&\quad + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{n} \sum_{r \in \mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} p^{(g)}_{0,rk}(n-l) - f^{(g)}_{0,\bar{p}} \pi^{(g)}_{0,k}| \leq \\
&\leq \sum_{i \in \mathbb{X}^{(g)}} p_i |p^{(g)}_{0,ik}(n) - \pi^{(g)}_{0,k}| + \\
&\quad + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{n-1} \sum_{r \in \mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} |p^{(g)}_{0,rk}(n-l) - \pi^{(g)}_{0,k}| + \\
&\quad + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{r \in \mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = n, X_{0,n} = r\} |\mathrm{I}(r = k) - \pi^{(g)}_{0,k}| + \\
&\quad + |\sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{n} \sum_{r \in \mathbb{X}^{(g)}} \mathsf{P}_i\{Z_0 = l, X_{0,l} = r\} - f^{(g)}_{0,\bar{d}}|\pi^{(g)}_{0,k} \leq \\
&\leq C_g \lambda_g^n + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{n-1} \mathsf{P}_i\{Z_0 = l\} C_g \lambda_g^{n-l} + \\
&\quad + \sum_{i \in \mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 = n\} + \sum_{i \in \mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 > n\} \leq \\
&\leq C_g \bar{\lambda}^n + \sum_{i \in \mathbb{X}^{(0)}} p_i \sum_{l=1}^{n-1} \mathsf{P}_i\{Z_0 = l\} \lambda_0^{-l} (\frac{\lambda_0}{\bar{\lambda}})^l C_g \bar{\lambda}^n + C_0 \bar{\lambda}^n \leq \\
&\leq (C_g + C_0 C_g + C_0) \bar{\lambda}^n = \bar{C} \bar{\lambda}^n.
\end{aligned} \tag{32}$$

Also, in this case, the following relation holds, for $n \geq 1, k \in \mathbb{X}^{(0)}$,

$$\begin{aligned}
\pi_{0,\bar{p},k}(n) &= \sum_{i \in \mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 > n, X_{0,n} = k\} \leq \\
&\leq \sum_{i \in \mathbb{X}^{(0)}} p_i \mathsf{P}_i\{Z_0 > n\} \leq C_0 \lambda_0^n \leq \bar{C}\bar{\lambda}^n.
\end{aligned} \tag{33}$$

The proof is complete. □

Lemma 2 implies that, in the singular case, where condition $\mathbf{B}_2$ hold and, thus the stationary distribution $\bar{\pi}_{0,\bar{p}} = \langle \pi_{0,\bar{p},j}, j \in \mathbb{X} \rangle$ depends on the initial distribution $\bar{p}$, the rate of convergence in the ergodic relation (17) is exponential, as it is in the regular case, where condition $\mathbf{A}_2$ holds.

The following theorem presents explicit upper bounds for deviations of stationary distributions of Markov chains $X_{\varepsilon,n}$ and $X_{0,n}$, for the singular perturbation model.

Theorem 3. *Let condition* $\mathbf{B}_2$ *holds. Then the following relation holds, for* $k \in \mathbb{X}$,

$$|\pi_{\varepsilon,k} - \pi_{0,\bar{d},k}| \leq \varepsilon(|d_k - \pi_{0,\bar{d},k}| + \frac{\bar{C}\bar{\lambda}}{1-\bar{\lambda}}). \tag{34}$$

Proof. Using relation (7) and (31) – (33), we get the following estimate, for $k \in \mathbb{X}$,

$$\begin{aligned}
|\pi_{\varepsilon,k} - \pi_{0,\bar{d},k}| &\leq |\varepsilon \sum_{l=0}^{\infty} p_{0,\bar{d},k}(l)(1-\varepsilon)^l - \pi_{0,\bar{d},k}| = \\
&= |\varepsilon \sum_{l=0}^{\infty} p_{0,\bar{d},k}(l)(1-\varepsilon)^l - \varepsilon \sum_{l=0}^{\infty} \pi_{0,\bar{d},k}(1-\varepsilon)^l| \leq \\
&\leq \varepsilon |d_k - \pi_{0,\bar{d},k}| + \varepsilon \sum_{l=1}^{\infty} \bar{C}\bar{\lambda}^l (1-\varepsilon)^l \leq \\
&\leq \varepsilon\left(|d_k - \pi_{0,\bar{d},k}| + \frac{\bar{C}\bar{\lambda}(1-\varepsilon)}{1-\bar{\lambda}(1-\varepsilon)}\right) \leq \varepsilon\left(|d_k - \pi_{0,\bar{d},k}| + \frac{\bar{C}\bar{\lambda}}{1-\bar{\lambda}}\right).
\end{aligned} \tag{35}$$

The proof is complete. □

4. Coupling and ergodic theorems for perturbed MCDC

In this section, we present coupling algorithms and get the effective upper bounds for the rate of convergence in ergodic theorems for regularly and singularly perturbed MCDC.

Let $\bar{p}' = \langle p'_i, i \in \mathbb{X}\rangle$ and $\bar{p}'' = \langle p''_1, i \in \mathbb{X}\rangle$ be two discrete probability distributions. Let us denote by $L[\bar{p}', \bar{p}'']$ the class of two-dimensional probability distribution $\bar{P} = \langle P_{ij}, (i,j) \in \mathbb{X} \times \mathbb{X}\rangle$ which satisfy the following conditions (a) $P'_i = \sum_{j \in \mathbb{X}} P_{ij} = p'_i$, $i \in \mathbb{X}$; (b) $P''_j = \sum_{i \in \mathbb{X}} P_{ij} = p''_j, j \in \mathbb{X}$.

Let us also denote,

$$Q_{\bar{P}} = \sum_{i \in \mathbb{X}} P_{ii} \quad \text{and} \quad Q(\bar{p}', \bar{p}'') = \sup_{\bar{P} \in L[\bar{p}', \bar{p}'']} Q_{\bar{P}}. \tag{36}$$

The following lemma presents the well known "coupling" result, which variants can be found in [23, 26, 32, 37] and [40–43, 47].

Lemma 3. *There exists the two-dimensional distribution* $\bar{P}^* = \langle P^*_{ij}, i, j \in \mathbb{X}\rangle \in L[\bar{p}', \bar{p}'']$ *such that*:

$$Q_{\bar{P}^*} = Q^* = \sum_{i \in \mathbb{X}} \min(p'_i, p''_i) = Q(\bar{p}', \bar{p}''). \tag{37}$$

The distribution $\bar{P}^*$ *is given by the following relations*:

(i) If $Q^* \in (0,1)$, *then*

$$\begin{aligned}
P^*_{ij} = &\min(p'_i, p''_j)\mathrm{I}(i=j) + \\
&+ \frac{1}{1-Q^*}(p'_i - \min(p'_i, p''_i))(p''_j - \min(p'_j, p''_j)), i, j \in \mathbb{X}.
\end{aligned} \tag{38}$$

(ii) $Q^* = 1$ *if and only if* $p'_k = p''_k, k \in \mathbb{X}$, *and*

$$P^*_{ij} = \min(p'_i, p''_j)\mathrm{I}(i=j),\ i, j \in \mathbb{X}. \tag{39}$$

(iii) $Q^* = 0$ *if and only if* $p'_k p''_k = 0, k \in \mathbb{X}$, *and*

$$P^*_{ij} = p'_i p''_j, i, j \in \mathbb{X}. \tag{40}$$

Proof. It can be found in the above mentioned works. In order, to improve self-readability of the present paper, we just give a short sketch of the proof. Obviously, probability $P_{ii} \leq p'_i \wedge p''_i, i \in \mathbb{X}$, for any two-dimensional distribution $\bar{P} = \langle P_{ij}, (i,j) \in \mathbb{X} \times \mathbb{X}\rangle \in \in L[\bar{p}', \bar{p}'']$. This relation implies that $Q_{\bar{P}} \leq Q^* = \sum_{i \in \mathbb{X}} p'_i \wedge p''_i$. This is easily to check that every relation (38), (39), or (40) defines a two-dimensional distribution $\bar{P}^*$ from the class $L[\bar{p}', \bar{p}'']$. Moreover, the corresponding quantity $Q_{\bar{P}^*} = Q^*$. This is obvious for two cases presented in propositions **(ii)** and **(iii)**. In the first case presented in proposition **(i)**, this follows from relation, $(p'_i - \min(p'_i, p''_i))(p''_i - \min(p'_i, p''_i)) = 0$, $i \in \mathbb{X}$. □

Let $\varepsilon \in (0,1]$. Let us us consider the random sequence $X^{(N)}_{\varepsilon,n} = X_{\varepsilon,Nn}, n = 0, 1, \ldots$, for some natural $N \geq 1$. It is a homogeneous Markov chain, with an initial distribution $\bar{p}$, the phase space $\mathbb{X}$, and the matrix of transition probabilities $\mathbf{P}^N_\varepsilon = \|p_{\varepsilon,ij}(N)\|$.

Let us define the quantities, for $i, j \in \mathbb{X}$,

$$Q^{(N)}_{\varepsilon,ij} = \sum_{r \in \mathbb{X}} \min(p_{\varepsilon,ir}(N), p_{\varepsilon,jr}(N)). \tag{41}$$

Let now use the multi-step coupling algorithm for construction a coupling Markov chain $Z^{(N)}_{\varepsilon,n} = (X'^{(N)}_{\varepsilon,n}, X''^{(N)}_{\varepsilon,n}), n = 0, 1, \ldots$, with:

(i) the phase space $\mathbb{Z} = \mathbb{X} \times \mathbb{X}$;

(ii) the initial distribution $\bar{P}_\varepsilon = \langle p_{\varepsilon,ij}, (i,j) \in \mathbb{Z}\rangle$ constructing according to relation (38), (39), or (40) for distributions $\bar{p}' = \bar{p} = \langle p_i, i \in \mathbb{X}\rangle$ and $\bar{p}'' = \bar{\pi}_\varepsilon = \langle \pi_{\varepsilon,i}, i \in \mathbb{X}\rangle$;

(iii) transition probabilities $P^{(N)}_{\varepsilon,ij,rk}$ defined by the following relations, for (i,j), $(r,k) \in \in \mathbb{Z}$:

(a) If $Q^{(N)}_{\varepsilon,ij} \in (0,1)$, then,

$$\begin{aligned} P^{(N)}_{\varepsilon,ij,rk} &= \mathsf{P}\{X'^{(N)}_{\varepsilon,1} = k, X''^{(N)}_{\varepsilon,1} = r / X'^{(N)}_{\varepsilon,0} = i, X''^{(N)}_{\varepsilon,0} = j\} = \\ &= \min(p_{\varepsilon,ir}(N), p_{\varepsilon,jk}(N))\mathrm{I}(r = k) + \\ &\quad + \frac{1}{1 - Q^{(N)}_{\varepsilon,ij}}(p_{\varepsilon,ir}(N) - \min(p_{\varepsilon,ir}(N), p_{\varepsilon,jr}(N))) \times \\ &\quad \times (p_{\varepsilon,jk}(N) - \min(p_{\varepsilon,ik}(N), p_{\varepsilon,jk}(N))), \end{aligned} \tag{42}$$

(b) If $Q^{(N)}_{\varepsilon,ij} = 1$, then $p_{\varepsilon,ir}(N) = p_{\varepsilon,jr}(N), r \in \mathbb{X}$, and

$$P^{(N)}_{\varepsilon,ij,rk} = \min(p_{\varepsilon,ir}(N), p_{\varepsilon,jk}(N))\mathrm{I}(r = k), r, k \in \mathbb{X}. \tag{43}$$

(c) If $Q^{(N)}_{\varepsilon,ij} = 0$, then $p_{\varepsilon,ir}(N)p_{\varepsilon,jr}(N) = 0, r \in \mathbb{X}$, and

$$P^{(N)}_{\varepsilon,ij,rk} = p_{\varepsilon,ir}(N)p_{\varepsilon,jk}(N), r, k \in \mathbb{X}. \tag{44}$$

The above construction of coupling Markov chain and the following lemma originate from works [23] and [37]. It plays an important role in what follows.

Lemma 4. *Let $Z^{(N)}_{\varepsilon,n} = (X'^{(N)}_{\varepsilon,n}, X''^{(N)}_{\varepsilon,n}), n = 0, 1, \ldots$, be a homogeneous Markov chain with the phase space $\mathbb{Z} = \mathbb{X} \times \mathbb{X}$, the initial distribution $\bar{P}_\varepsilon$ and transition probabilities given by relations* (42)–(44)*. Then*:

(i) *The first component, $X'^{(N)}_{\varepsilon,n}, n = 0, 1, \ldots$, is a homogeneous Markov chain with the phase space $\mathbb{X}$, the initial distribution $\bar{p}$ and the matrix of transition probabilities $\mathbf{P}^N_\varepsilon$.*

(ii) *The second component $X''^{(N)}_{\varepsilon,n}, n = 0, 1, \ldots$ is a homogeneous Markov chain with the phase space $\mathbb{X}$, the initial distribution $\bar{\pi}_\varepsilon$ and the matrix of transition probabilities $\mathbf{P}^N_\varepsilon$.*

(iii) *The set $\mathbb{Z}_0 = \{(i,i), i \in \mathbb{X}\}$ is an absorbing set for the Markov chain $Z^{(N)}_{\varepsilon,n}$, i.e., probabilities $P^{(N)}_{\varepsilon,ii,rk} = 0$, for $i, r, k \in \mathbb{X}, r \neq k$.*

We also refer to preprint [3], where the proof of Lemma 4 can be found.

Let $\mathbf{A} = \|a_{ij}\|$ be a $m \times m$ a matrix with real-valued elements. Let us introduce functional,

$$Q(\mathbf{A}) = \min_{1 \le i,j \le m} \sum_{k=1}^{m} a_{ik} \wedge a_{jk}. \tag{45}$$

The following simple lemma presents some basic properties of functional $Q(\mathbf{A})$.

Lemma 5. *Functional $Q(\mathbf{A})$ possesses the following properties*: (a) $Q(a\mathbf{A}) = aQ(\mathbf{A})$, *for any* $a \ge 0$; (b) $Q(\mathbf{A}) \ge a_1 Q(\mathbf{A}_1) + \cdots + a_n Q(\mathbf{A}_n)$, *for any* $m \times m$ *matrices* $\mathbf{A}_1, \dots, \mathbf{A}_n$ *with real-valued elements, numbers* $a_1, \dots, a_n \ge 0, a_1 \cdots + a_n = 1$, *and matrix* $\mathbf{A} = a_1\mathbf{A}_1 + \cdots + a_n\mathbf{A}_n$, *for* $n \ge 2$; (c) $Q(\mathbf{A}) \in [0, 1]$, *for any stochastic matrix* $\mathbf{A}$; (d) $Q(\mathbf{A}) = 1$, *for any* $m \times m$ *stochastic damping type matrix* $\mathbf{A} = \|a_{ij}\|$, *with elements* $a_{ij} = a_j \ge 0, i, j = 1, \dots, m$.

The following useful proposition takes place.

Lemma 6. *The following inequality takes place, for* $N \ge 1$ *and* $\varepsilon \in (0, 1]$,

$$1 - Q(\mathbf{P}_\varepsilon^N) \le (1 - Q(\mathbf{P}_0^N))(1 - \varepsilon)^N. \tag{46}$$

Proof. Relation, $\mathbf{AB} = \mathbf{B}$, holds for any $m \times m$ stochastic matrix $\mathbf{A} = \|a_{ij}\|$ and $m \times m$ stochastic damping type matrix $\mathbf{B} = \|b_{ij}\|$, with elements $b_{ij} = b_j \ge 0, i, j = 1, \dots, m$. Also, matrix $\mathbf{C} = \mathbf{BA}$, which has elements, $c_{ij} = c_j = \sum_{k=1}^{m} b_k a_{kj} \ge 0, i, j = 1, \dots, m$, is a stochastic damping type matrix, i.e., it has all rows the same.

Using these remarks, we get the following relation, for $N \ge 1$,

$$\begin{aligned} \mathbf{P}_\varepsilon^N &= ((1 - \varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D})^N = \mathbf{P}_\varepsilon^{N-1}(1 - \varepsilon)\mathbf{P}_0 + \mathbf{P}_\varepsilon^{N-1}\varepsilon\mathbf{D} = \mathbf{P}_\varepsilon^{N-1}(1 - \varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D} = \\ &= \mathbf{P}_\varepsilon^{N-2}(1 - \varepsilon)^2\mathbf{P}_0^2 + \mathbf{P}_\varepsilon^{N-2}\varepsilon(1 - \varepsilon)\mathbf{D}\mathbf{P}_0 + \varepsilon\mathbf{D} = \\ &= \cdots = (1 - \varepsilon)^N\mathbf{P}_0^N + \varepsilon(1 - \varepsilon)^{N-1}\mathbf{D}\mathbf{P}_0^{N-1} + \cdots + \varepsilon\mathbf{D}. \end{aligned} \tag{47}$$

Using relation (47) and Lemma 5, we get the following relation,

$$\begin{aligned} Q(\mathbf{P}_\varepsilon^N) &\ge (1 - \varepsilon)^N Q(\mathbf{P}_0^N) + \varepsilon(1 - \varepsilon)^{N-1} Q(\mathbf{D}\mathbf{P}_0^{N-1}) + \cdots + \varepsilon Q(\mathbf{D}) = \\ &= (1 - \varepsilon)^N Q(\mathbf{P}_0^N) + \varepsilon(1 - \varepsilon)^{N-1} + \cdots + \varepsilon = \\ &= (1 - \varepsilon)^N Q(\mathbf{P}_0^N) + 1 - (1 - \varepsilon)^N. \end{aligned} \tag{48}$$

This relation is equivalent to inequality (46). □

Let us introduce, for $N \ge 1$, the coefficient of ergodicity,

$$\Delta_N(\mathbf{P}_0) = (1 - Q(\mathbf{P}_0^N))^{1/N}. \tag{49}$$

The given below Theorems 4 and 5 present effective coupling type upper bounds for the rate of convergence in the individual ergodic theorem for MCDC. These theorems are based on corresponding general coupling results for Markov chains given in [23, 32, 37] and specify and detail the corresponding coupling upper bounds for the rate of convergence in ergodic theorems for MCDC.

Note that neither condition $\mathbf{A}_1$ nor condition $\mathbf{B}_1$ is required in Theorem 4 formulated below.

Also, we count $\Delta_N(\mathbf{P}_0)^0 = 1$, if $\Delta_N(\mathbf{P}_0) = 0$.

Theorem 4. *The following relation takes place, for every* $\bar{p} \in L_m$, $j \in \mathbb{X}$, $n \ge 0$, *and* $\varepsilon \in (0, 1]$,

$$|p_{\varepsilon,\bar{p},j}(n) - \pi_{\varepsilon,j}| \le (1 - Q(\bar{p}, \bar{\pi}_\varepsilon))\Delta_N(\mathbf{P}_0)^{[n/N]N}(1 - \varepsilon)^{[n/N]N}. \tag{50}$$

Proof. Since, the initial distribution of Markov chain $X''^{(N)}_{\varepsilon,n}$ coincides with its stationary distribution, this Markov chain is a stationary random sequence and, thus, for $j \in \mathbb{X}$, $n \geq 0$,

$$\mathsf{P}\{X''^{(N)}_{\varepsilon,n} = j\} = \pi_{\varepsilon,j}. \tag{51}$$

Let us now define the hitting (coupling) time,

$$T^{(N)}_{\varepsilon} = \min(n \geq 0 : X'^{(N)}_{\varepsilon,n} = X''^{(N)}_{\varepsilon,n}) = \min(n \geq 0 : Z^{(N)}_{\varepsilon,n} \in \mathbb{Z}_0). \tag{52}$$

Since $\mathbb{Z}_0$ is an absorbing set for the Markov chain $Z^{(N)}_{\varepsilon,n}$, the following relation holds,

$$\mathsf{P}\{Z^{(N)}_{\varepsilon,n} \in \mathbb{Z}_0, n \geq T^{(N)}_{\varepsilon}\} = 1. \tag{53}$$

Using the above remarks, we get the following relation, for $j \in \mathbb{X}, n \geq 0$,

$$\begin{aligned}|p_{\varepsilon,\bar{p},j}(Nn) - \pi_{\varepsilon,j}| &= |\mathsf{P}\{X'^{(N)}_{\varepsilon,n} = j\} - \mathsf{P}\{X''^{(N)}_{\varepsilon,n} = j\}| = \\ &= |\mathsf{P}\{X'^{(N)}_{\varepsilon,n} = j, X''^{(N)}_{\varepsilon,n} \neq j\} - \mathsf{P}\{X'^{(N)}_{\varepsilon,n} \neq j, X''^{(N)}_{\varepsilon,n} = j\}| \leq \\ &\leq \mathsf{P}\{X'^{(N)}_{\varepsilon,n} = j, X''^{(N)}_{\varepsilon,n} \neq j\} + \mathsf{P}\{X'^{(N)}_{\varepsilon,n} \neq j, X''^{(N)}_{\varepsilon,n} = j\} \leq \\ &\leq \mathsf{P}\{T^{(N)}_{\varepsilon} > n\}.\end{aligned} \tag{54}$$

Using Lemma 6, we get, for $\bar{p} \in L_m, j \in \mathbb{X}$,

$$\begin{aligned}|p_{\varepsilon,\bar{p},j}(0) - \pi_{\varepsilon,j}| &\leq \mathsf{P}\{X'^{(N)}_{\varepsilon,0} = j, X''^{(N)}_{\varepsilon,0} \neq j\} + \mathsf{P}\{X'^{(N)}_{\varepsilon,0} \neq j, X''^{(N)}_{\varepsilon,0} = j\} \leq \\ &\leq \mathsf{P}\{T^{(N)}_{\varepsilon} > 0\} = 1 - Q(\bar{p}, \bar{\pi}_{\varepsilon}).\end{aligned} \tag{55}$$

Also, by continuing inequality (54), we get, for $\bar{p} \in L_m, j \in \mathbb{X}, n \geq 0$,

$$\begin{aligned}|p_{\varepsilon,\bar{p},j}&(Nn) - \pi_{\varepsilon,j}| \leq \mathsf{P}\{T^{(N)}_{\varepsilon} > n\} = \\ &= \sum_{i,j \in \mathbb{X}} \mathsf{P}\{X'^{(N)}_{\varepsilon,n} \neq X''^{(N)}_{\varepsilon,n} / X'^{(N)}_{\varepsilon,n-1} = i, X''^{(N)}_{\varepsilon,n-1} = j\} \times \\ &\quad \times \mathsf{P}\{T^{(N)}_{\varepsilon} > n-1, X'^{(N)}_{\varepsilon,n-1} = i, X''^{(N)}_{\varepsilon,n-1} = j\} = \\ &= \sum_{i,j \in \mathbb{X}} \mathsf{P}\{T^{(N)}_{\varepsilon} > n-1, X'^{(N)}_{\varepsilon,n-1} = i, X''^{(N)}_{\varepsilon,n-1} = j\}(1 - Q^{(N)}_{\varepsilon,ij}) \leq \\ &\leq \mathsf{P}\{T^{(N)}_{\varepsilon} > n-1\}(1 - Q(\mathbf{P}^N_{\varepsilon})) \leq \\ &\leq \cdots \leq \mathsf{P}\{T_{\varepsilon} > 0\}(1 - Q(\mathbf{P}^N_{\varepsilon}))^n \leq \\ &\leq (1 - Q(\bar{p}, \bar{\pi}_{\varepsilon}))(1 - Q(\mathbf{P}^N_0))^n (1 - \varepsilon)^{Nn} = \\ &= (1 - Q(\bar{p}, \bar{\pi}_{\varepsilon}))\Delta_N(\mathbf{P}_0))^{Nn}(1 - \varepsilon)^{Nn}.\end{aligned} \tag{56}$$

Also, for $\bar{p} \in L_m, j \in \mathbb{X}, n \geq 0$ and $l = 0, \ldots, N-1$,

$$\begin{aligned}|p_{\varepsilon,\bar{p},j}&(Nn + l) - \pi_{\varepsilon,j}| = |\sum_{k \in \mathbb{X}} p_{\varepsilon,\bar{p},k}(nN)p_{\varepsilon,kj}(l) - \sum_{k \in \mathbb{X}} \pi_{\varepsilon,k}p_{\varepsilon,kj}(l)| \leq \\ &\leq \sum_{k \in \mathbb{X}} |p_{\varepsilon,\bar{p},k}(Nn) - \pi_{\varepsilon,k}|p_{\varepsilon,kj}(l) \leq \\ &\leq \max_{k \in \mathbb{X}} |p_{\varepsilon,\bar{p},k}(Nn) - \pi_{\varepsilon,k}| \leq \\ &\leq (1 - Q(\bar{p}, \bar{\pi}_{\varepsilon}))\Delta_N(\mathbf{P}_0)^{Nn}(1 - \varepsilon)^{Nn}.\end{aligned} \tag{57}$$

Inequalities (56) and (57) imply inequalities given in relation (50). The proof is complete. □

The upper bounds given in relation (50) become better if quantities $1 - Q(\bar{p}, \bar{\pi}_{\varepsilon})$, $\Delta_N(\mathbf{P}_0)$ and $1 - \varepsilon$ take smaller values. The factor $1 - Q(\bar{p}, \bar{\pi}_{\varepsilon})$, is determined by a prior information about the stationary probabilities. It takes smaller values if one can choose initial distribution $\bar{p}$ with smaller deviation from the stationary distribution $\bar{\pi}_\varepsilon$. Relation (50) gives an effective upper bounds for the rate of convergence in the corresponding individual ergodic theorem for the Markov chain $X_{\varepsilon,n}$ even in the case, where factor $\Delta_N(\mathbf{P}_0) = 1$.

It also worth noting that the weaker upper bound $(1-\varepsilon)^n$ on the right hand side of inequality (50) have been given for Markov chains with a general phase and damping component, in the recent paper [6].

In the case, where condition $\mathbf{A}_1$ holds (i.e., the phase space $\mathbb{X}$ is one aperiodic class of communicative states for the Markov chain $X_{0,n}$), $1 - Q(\mathbf{P}_0^N) \to 0$ as $N \to \infty$, and, thus, the following condition holds for N large enough:

$\mathbf{C}_N$: $\Delta_N(\mathbf{P}_0) < 1$.

Also, condition $\mathbf{C}_N$ is, for every $N \geq 1$, sufficient for holding the mentioned above weaken variant of condition $\mathbf{A}_2$.

Indeed, probabilities $p_{\varepsilon,\bar{p},j}(n) \to p_{0,\bar{p},j}(n)$ as $\varepsilon \to 0$, for any $j \in \mathbb{X}, n \geq 0$. Since stationary probabilities $\pi_{\varepsilon,j} \in [0,1], j \in \mathbb{X}$, any sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $0 < \varepsilon_{n_l} \to 0$ as $l \to \infty$ such that $\pi_{\varepsilon_{n_l},j} \to \pi_{0,j}$ as $l \to \infty$, for $j \in \mathbb{X}$. By passing $\varepsilon \to 0$ in the inequality (50), we get the following relation holding for $\bar{p} \in L_m, j \in \mathbb{X}, n \geq 0$,

$$
\begin{aligned}
|p_{0,\bar{p},j}(n) - \pi_{0,j}| &\leq (1 - Q(\bar{p}, \bar{\pi}_0))\Delta_N(\mathbf{P}_0)^{[n/N]N} \\
&\leq \Delta_N(\mathbf{P}_0)^{-N}\Delta_N(\mathbf{P}_0)^n = (1 - Q_N(\mathbf{P}_0^N))^{-1}\Delta_N(\mathbf{P}_0)^n, \qquad (58)
\end{aligned}
$$

where one should count $\Delta_N(\mathbf{P}_0)^0 = 1$, if $\Delta_N(\mathbf{P}_0) = 0$.

Relation (58) obviously implies that $p_{0,\bar{p},j}(n) \to \pi_{0,j}$ as $n \to \infty$, for $j \in \mathbb{X}$. Thus, limits $\pi_{0,j}, j \in \mathbb{X}$ are the same for any subsequences ε_n and ε_{n_l} and, thus, stationary probabilities $\pi_{\varepsilon,j} \to \pi_{0,j}$ as $\varepsilon \to 0$, for $j \in \mathbb{X}$. By derecting $\varepsilon \to 0$ in the equations given in relation (10), we get that limits $\pi_{0,j}, j \in \mathbb{X}$ satisfy the system of linear equations (10) and, thus, $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X} \rangle$ is the stationary distribution for the Markov chain $X_{0,n}$. Some components of this stationary distribution can be equal 0. In this case, set $\mathbb{X}_1 = \{j \in \mathbb{X} : \pi_{0,j} > 0\}$ is a closed, aperiodic class of communicative states, while set $\mathbb{X}_0 = \{j \in \mathbb{X} : \pi_{0,j} = 0\}$ is the class of transient states, for the Markov chain $X_{0,n}$.

If the stationary distribution $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X} \rangle$ is positive, then $\mathbb{X}_0 = \emptyset$. In this case, condition $\mathbf{C}_N$ is sufficient for holding of condition $\mathbf{A}_1$.

Further, relation (58) implies that the Markov chain $X_{0,n}$ is ergodic with an exponential rate of convergence in the corresponding ergodic theorem, if condition $\mathbf{C}_N$ holds, for some $N \geq 1$.

In the case, where $\mathbf{A}_1$ and some minor technical conditions hold, $\Delta_N\big(\mathbf{P}_0\big) \to |\rho_{1,2}|$ as $N \to \infty$, where $\rho_{1,2}$ is the second eigenvalue for matrix $\mathbf{P}_0$. The corresponding comments can be found in [3]. The above asymptotic relation show that the coupling upper bounds for rate of convergence in individual ergodic relations given in Theorem 4 usually are asymptotically equivalent with analogous upper bounds, which can be obtained with the use of eigenvalue decomposition representation for transition probabilities. At the same time, computing of coefficients of ergodicity $\Delta_N(\mathbf{P}_0)$ does not require solving of the polynomial equation, $\det(\rho\mathbf{I} - \mathbf{P}_0) = 0$, that is required for finding eigenvalues.

Let us assume that the following condition holds for some $N \geq 1$ and $h > 1$:

$\mathbf{D}_N$: The phase space $\mathbb{X} = \cup_{g=0}^h \mathbb{X}^{(g)}$, where: **(a)** $\mathbb{X}^{(g)}$, $g = 1, \ldots, h$, are non-intersecting subsets of $\mathbb{X}$, **(b)** $\mathbb{X}^{(g)}, g = 1, \ldots, h$, are non-empty, closed classes of states for the Markov chain $X_{0,n}$, **(c)** $\Delta_N(\mathbf{P}_{0,g}) < 1$, for $g = 1, \ldots, h$, **(d)** $X^{(0)}$ is a class of states for the Markov chain $X_{0,n}$ such that relation (27) holds (if $X^{(0)}$ is a non-empty set).

Remark 1. Condition $\mathbf{D}_N$ implies that condition $\mathbf{B}_2$ holds, with constants $C_0 = C_{0,N}$ and $\lambda_0 = \lambda_{0,N}$ appearing in relation (27) and constants $C_g = C_{g,N} = \Delta_N(\mathbf{P}_{0,g})^{-N}$, $\lambda_g = \lambda_{g,N} = \Delta_N(\mathbf{P}_{0,g}), g = 1, \dots, h$ given by inequalities (58). Thus, Lemma 2 takes place, with constants $\bar{C} = \bar{C}_N$ and $\bar{\lambda} = \bar{\lambda}_N$ given by relation (28) and (29), in which constants $C_g = C_{g,N}, \lambda_g = \lambda_{g,N}, g = 0, \dots, h$.

It is useful also to note that condition $\mathbf{B}_2$ implies that condition $\mathbf{D}_N$ holds for N large enough.

Let us denote, for $\bar{p}', \bar{p}'' \in L_m, k \in \mathbb{X}$,

$$\delta_{0,\bar{p}',\bar{p}'',k} = |\pi_{0,\bar{p}',k} - \pi_{0,\bar{p}'',k}| = \\ = \begin{cases} |f^{(g)}_{0,\bar{p}'} - f^{(g)}_{0,\bar{p}''}|\pi^{(g)}_{0,k} & \text{for } k \in \mathbb{X}^{(g)}, g = 1, \dots, h, \\ 0 & \text{for } k \in \mathbb{X}^{(0)}. \end{cases} \tag{59}$$

Theorem 5. *Let condition $\mathbf{D}_N$ holds. Then, the following relation takes place, for $\bar{p} \in L_m$, $k \in \mathbb{X}^{(g)}, g = 0, \dots, h$, $n \geq 0$, and $\varepsilon \in (0, 1]$,*

$$|p_{\varepsilon,\bar{p},k}(n) - \pi_{\varepsilon,k}| \leq \big(2\bar{C}_N\bar{\lambda}^n_N + \delta_{0,\bar{p},\bar{\pi}_\varepsilon,k}\big)(1-\varepsilon)^n \leq \\ \leq \big(2\bar{C}_N\bar{\lambda}^n_N + 1\big)(1-\varepsilon)^n. \tag{60}$$

Proof. By using the renewal type relation (6), condition $\mathbf{D}_N$, and taking into account stationarity of the Markov chain $X_{\varepsilon,n}$, with the initial distribution $\bar{\pi}_\varepsilon$, we get the following relation, for $k \in \mathbb{X}^{(g)}, g = 1, \dots, h$, and $n \geq 0$,

$$|p_{\varepsilon,\bar{p},k}(n) - \pi_{\varepsilon,k}| = |p_{\varepsilon,\bar{p},k}(n) - p_{\varepsilon,\bar{\pi}_\varepsilon,k}(n)| = \\ = |p_{0,\bar{p},k}(n) - p_{0,\bar{\pi}_\varepsilon,k}(n)|(1-\varepsilon)^n. \tag{61}$$

Lemmas 1 and 2, let us continue relation (61),

$$|p_{0,\bar{p},k}(n) - p_{0,\bar{\pi}_\varepsilon,k}(n)|(1-\varepsilon)^n \leq \big(|p_{0,\bar{p},k}(n) - \pi_{0,\bar{p},k}| + |p_{0,\bar{\pi}_\varepsilon,k}(n) - \pi_{0,\bar{\pi}_\varepsilon,k}| + \\ + |\pi_{0,\bar{p},k} - \pi_{0,\bar{\pi}_\varepsilon,k}|\big)(1-\varepsilon)^n \leq \\ \leq \big(2\bar{C}_N\bar{\lambda}^n_N + \delta_{0,\bar{p},\bar{\pi}_\varepsilon,k}\big)(1-\varepsilon)^n. \tag{62}$$

Relations (61) and (62) imply relation (60) to hold. □

5. Ergodic theorems for perturbed MCDC

In his section we present ergodic theorems for regularly and singularly perturbed MCDC.

First, let us consider the case of regularly perturbed MCDC. The following theorem takes place.

Theorem 6. *Let condition $\mathbf{C}_N$ holds for some $N \geq 1$. Then, for $\bar{p} \in L_m, k \in \mathbb{X}$ and any $n_\varepsilon \to \infty$ as $\varepsilon \to 0$,*

$$p_{\varepsilon,\bar{p},k}(n_\varepsilon) \to \pi_{0,k} = \pi_{0,\bar{d},k} \text{ as } \varepsilon \to 0. \tag{63}$$

Proof. Using the renewal type relation (6) and inequality (58), we get that the following relation holds, for $k \in \mathbb{X}$ and any $n_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$|p_{\varepsilon,\bar{p},k}(n_\varepsilon) - \pi_{0,k}| = |p_{\varepsilon,\bar{p},k}(n_\varepsilon) - \pi_{0,k}(1-\varepsilon)^{n_\varepsilon} - \pi_{0,k}(1-(1-\varepsilon)^{n_\varepsilon})| = \\ = |(p_{0,\bar{p},k}(n_\varepsilon) - \pi_{0,k})(1-\varepsilon)^{n_\varepsilon} + \sum_{l=1}^{n_\varepsilon}(p_{0,\bar{d},k}(n_\varepsilon - l) - \pi_{0,k})\varepsilon(1-\varepsilon)^{l-1}| \leq \\ \leq (1 - Q(\bar{p}, \bar{\pi}_0))\Delta_N(\mathbf{P}_0)^{[n_\varepsilon/N]N} + (1 - Q(\bar{d}, \bar{\pi}_0))\varepsilon\sum_{l=1}^{n_\varepsilon}\Delta_N(\mathbf{P}_0)^{[(n_\varepsilon - l)/N]N} \leq$$

$$\leq (1 - Q(\bar{p}, \bar{\pi}_0))\Delta_N(\mathbf{P}_0)^{[n_\varepsilon/N]N} + (1 - Q(\bar{d}, \bar{\pi}_0))N(1 - \Delta_N(\mathbf{P}_0)^N)^{-1}\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{64}$$

The proof is complete. □

Remark 2. Relation (64) gives, in fact, explicit upper bounds for the rate of convergence in the ergodic relation given in Theorem 7.

Ergodic theorems for singularly perturbed MCDC take much more complex forms.

Theorem 7. *Let condition* $\mathbf{D}_N$ *holds for some* $N \geq 1$*. Then the following ergodic relations take place, for* $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \dots, h$:

(i) *If* $n_\varepsilon \to \infty$ *and* $\varepsilon n_\varepsilon \to \infty$ *as* $\varepsilon \to 0$*, then,*

$$p_{\varepsilon,\bar{p},k}(n_\varepsilon) \to \pi_{0,\bar{p},k}(\infty) = \pi_{0,\bar{d},k} \text{ as } \varepsilon \to 0. \tag{65}$$

(ii) *If* $n_\varepsilon \to \infty$ *and* $\varepsilon n_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0$*, then,*

$$p_{\varepsilon,\bar{p},k}(n_\varepsilon) \to \pi_{0,\bar{p},k}(t) = \pi_{0,\bar{p},k}e^{-t} + \pi_{0,\bar{d},k}(1 - e^{-t}) \text{ as } \varepsilon \to 0. \tag{66}$$

(iii) *If* $n_\varepsilon \to \infty$ *and* $\varepsilon n_\varepsilon \to 0$ *as* $\varepsilon \to 0$*, then,*

$$p_{\varepsilon,\bar{p},k}(n_\varepsilon) \to \pi_{0,\bar{p},k}(0) = \pi_{0,\bar{p},k} \text{ as } \varepsilon \to 0. \tag{67}$$

Proof. The renewal type relation (6) written for $n = n_\varepsilon$ takes the following form, for $\bar{p} \in L_m, k \in \mathbb{X}$,

$$p_{\varepsilon,\bar{p},k}(n_\varepsilon) = p_{0,\bar{p},k}(n_\varepsilon)(1 - \varepsilon)^{n_\varepsilon} + \sum_{l=1}^{n_\varepsilon} p_{0,\bar{d},k}(n_\varepsilon - l)\varepsilon(1 - \varepsilon)^{l-1}. \tag{68}$$

By applying inequality (34) given in Lemma 2, with constants $\bar{C}_N, \bar{\lambda}_N$ pointed in Remark 1, to the transition probabilities appearing in the above renewal type relation, we get the following inequality, $\bar{p} \in L_m, k \in \mathbb{X}^{(g)}, g = 1, \dots, h$,

$$\begin{aligned} &|p_{\varepsilon,\bar{p},k}(n_\varepsilon) - \pi_{0,\bar{p},k}(1 - \varepsilon)^{n_\varepsilon} - \pi_{0,\bar{d},k}(1 - (1 - \varepsilon)^{n_\varepsilon})| = \\ &\quad = |(p_{0,\bar{p},k}(n_\varepsilon) - \pi_{0,\bar{p},k})(1 - \varepsilon)^{n_\varepsilon} + \sum_{l=1}^{n_\varepsilon}(p_{0,\bar{d},k}(n_\varepsilon - l) - \pi_{0,\bar{d},k})\varepsilon(1 - \varepsilon)^{l-1}| \leq \\ &\quad \leq \bar{C}_N\bar{\lambda}_N^{n_\varepsilon} + \varepsilon\sum_{l=1}^{n_\varepsilon}\bar{C}_N\bar{\lambda}_N^{[(n_\varepsilon - l)/N]N} \leq \\ &\quad \leq \bar{C}_N\bar{\lambda}_N^{n_\varepsilon} + N(1 - \bar{\lambda}_N^N)^{-1}\varepsilon. \end{aligned} \tag{69}$$

Let us introduce function $R_\varepsilon(t) = |(1 - \varepsilon)^{n_\varepsilon} - e^{-t}|, t \in [0, \infty]$.

If $n_\varepsilon \to \infty$ and $\varepsilon n_\varepsilon \to t \in [0, \infty]$ as $\varepsilon \to 0$, then,

$$R_\varepsilon(t) \to 0 \text{ as } \varepsilon \to 0. \tag{70}$$

The following inequality takes place,

$$\begin{aligned} &|\pi_{0,\bar{p},k}e^{-t} + \pi_{0,\bar{d},k}(1 - e^{-t}) - \pi_{0,\bar{p},k}(1 - \varepsilon)^{n_\varepsilon} - \pi_{0,\bar{d},k}(1 - (1 - \varepsilon)^{n_\varepsilon})| \leq \\ &\qquad \leq |\pi_{0,\bar{p},k} - \pi_{0,\bar{d},k}|R_\varepsilon(t) = \delta_{0,\bar{p},\bar{d},k}R_\varepsilon(t). \end{aligned} \tag{71}$$

Relations (69)–(71) obviously imply that the following relation holds, for $k \in \mathbb{X}^{(g)}$, $k = 0, \dots, h$, if $n_\varepsilon \to \infty$ and $\varepsilon n_\varepsilon \to t \in [0, \infty]$ as $\varepsilon \to 0$,

$$|p_{\varepsilon,\bar{p},k}(n_\varepsilon) - \pi_{0,\bar{p},k}(t)| \leq \bar{C}_N\bar{\lambda}_N^{n_\varepsilon} + N(1 - \bar{\lambda}_N^N)^{-1}\varepsilon + \delta_{0,\bar{p},\bar{d},k}R_\varepsilon(t) \to 0 \text{ as } \varepsilon \to 0. \tag{72}$$

This relation proves the theorem. □

Remark 3. Inequality (72) gives, in fact, explicit upper bounds for the rate of convergence in ergodic relation given in Theorem 8. Of course, it is possible to get some simple explicit upper bounds for $R_\varepsilon(t)$ in terms of quantities $\varepsilon n_\varepsilon$ and t.

6. Conclusion

One of the main reasons for approximation of the Markov chain $X_{0,n}$ (with the matrix of transition probabilities $\mathbf{P}_0$), by perturbed (regularised) MCDC $X_{\varepsilon,n}$ (with the matrix of transition probabilities $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$), is to use it for approximation of the stationary distribution $\bar{\pi}_0 = \langle \pi_{0,j}, j \in \mathbb{X}\rangle$ of the Markov chain $X_{0,n}$. Since the corresponding phase space $\mathbb{X} = \{1,\dots,m\}$ can be large, the power method can be used for approximative computing of stationary distribution $\bar{\pi}_0$. In this case, its components $\pi_{0,j}$ are approximated by probabilities $p_{\varepsilon,\bar{p},j}(n) = \sum_{i\in\mathbb{X}} p_i p_{\varepsilon,ij}(n)$, where $p_{\varepsilon,ij}(n)$ are elements of the matrix $\mathbf{P}_\varepsilon^n$ and $\bar{p} = \langle p_j, j \in \mathbb{X}\rangle$ is some initial distribution.

The results given in Theorems 7 and 8 show that the situation significantly differ for two models: (a) regular, where the phase space $\mathbb{X}$ is one class of communicative states for the Markov chain $X_{0,n}$ (condition $\mathbf{A}_1$ holds) and, (b) singular, where the phase space $\mathbb{X}$ splits in several closed classes of communicative states for the Markov chain $X_{0,n}$ (condition $\mathbf{B}_1$ holds, for some $h > 1$).

In the regular case, Theorem 7 shows that one can approximate the stationary probabilities $\pi_{0,j}$ by probabilities $p_{\varepsilon,\bar{p},j}(n_\varepsilon)$, using arbitrary positive integers $n_\varepsilon \to \infty$ as $\varepsilon \to 0$. Moreover, the explicit upper bounds for $|p_{\varepsilon,\bar{p},j}(n_\varepsilon) - \pi_{0,j}|$ pointed out in Remark 2 let one balance the choice of ε and n_ε.

In the singular case, the situation is more complex. If $\bar{p} \neq \bar{d}$, one should be more careful, since in this case it may be that the stationary probability $\pi_{0,\bar{d},k} \neq \pi_{0,\bar{p},k}$. In this case, Theorem 8 answers the question about applicability probabilities $p_{\varepsilon,\bar{p},j}(n_\varepsilon)$ as approximations for stationary probabilities for the Markov chain $X_{0,n}$. In fact, these probabilities converge to some mixture of stationary probabilities $\pi_{0,\bar{p},k}$ and $\pi_{0,\bar{d},k}$, namely, $\pi_{0,\bar{p},k}(t) = \pi_{0,\bar{p},k}e^{-t} + \pi_{0,\bar{d},k}(1-e^{-t})$, as $n_\varepsilon \to \infty$ in such way that $\varepsilon n_\varepsilon \to t \in [0,\infty]$ as $\varepsilon \to 0$. Moreover, the explicit upper bounds for $|p_{\varepsilon,\bar{p},k}(n_\varepsilon) - \pi_{0,\bar{p},k}(t)|$ pointed out in Remark 3 let one also balance the choice of ε and n_ε and, in some sense, predict the value of limit $\pi_{0,\bar{p},k}(t)$ depending on the value of quantity $\varepsilon n_\varepsilon$.

We would like also to note that some experimental numerical results supporting the theoretical results presented in the present paper can be found in preprint [3], where one can also find a detailed survey of works related to applications of Markov chains with damping components.

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КАПЛІНГ І ЕРГОДИЧНІ ТЕОРЕМИ ДЛЯ ЛАНЦЮГІВ МАРКОВА З ДЕМПФУЮЧОЮ КОМПОНЕНТОЮ

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Анотація. Збурені ланцюги Маркова є популярними моделями для опису інформаційних мереж. У таких моделях матриця переходу $\mathbf{P}_0$ інформаційного ланцюга Маркова звичайно апроксимується матрицею $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$, де $\mathbf{D}$ — так звана демпфуюча стохастична матриця з однаковими рядками і додатними елементами, а $\varepsilon \in [0,1]$ — параметр демпфування (збурення). Використовуючи процедуру штучної регенерації для збуреного ланцюга Маркова $\eta_{\varepsilon,n}$ з матрицею ймовірностей переходів $\mathbf{P}_\varepsilon$ і каплінг методи, ми отримуємо ергодичні теореми у вигляді асимптотичних співвідношень для $p_{\varepsilon,ij}(n) = \mathsf{P}_i\{\eta_{\varepsilon,n} = j\}$ при $n \to \infty$ і $\varepsilon \to 0$, а також явні верхні оцінки швидкості збіжності в таких теоремах. Зокрема, досліджується найбільш складний випадок моделей із сингулярними збуреннями, коли фазовий простір незбуреного ланцюга Маркова $\eta_{0,n}$ розщеплюється на кілька замкнутих класів комунікативних станів і, можливо, клас перехідних станів.