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TESTING HYPOTHESES FOR MEASURES WITH DIFFERENT MASSES: FOUR OPTIMIZATION PROBLEMS

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Abstract. We consider a problem similar to testing two composite hypotheses, where measures constituting the hypotheses are not probabilities and may have different masses. Then it is naturally to consider four different optimization problems. To characterize optimal solutions we introduce corresponding dual optimization problems. Our main goal is to find sufficient conditions for the existence of saddle points in each problem.

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1. Introduction

The problem of testing two composite hypotheses in statistics has a long history. Here we briefly mention some key papers. If the hypotheses are simple, the fundamental Neyman–Pearson lemma describes a test which minimizes the second type error (or maximizes the power) among all tests of given level. A classical result of Lehmann [10, Section 8.1] gives sufficient conditions under which the Neyman–Pearson test for Bayesian mixtures is a maximin test for composite hypotheses. However, it says nothing how to find such mixtures. Moreover, they may not exist. Krafft and Witting [9] introduced a dual problem, whose solution, if it exists, provides required mixtures. Baumann [1] introduced a dual problem in the space of finitely additive set functions. This problem always has a solution, which allows us to give a complete characterization of maximin tests, but it has only theoretical value, since it involves finitely additive set functions. Cvitanić and Karatzas [2] extended the domain of the definition of the dual problem introduced by Krafft and Witting and thus weakened the assumptions under which the dual problem has a solution. Their result was generalized by Gushchin [6] where the reader can find a more detailed history of the problem and a discussion concerning relations between different existing results.

At the same time in applications, especially in mathematical finance, there often appear optimization problems of the same type but where the hypothesis and alternative contain (nonnegative and finite) measures that are not assumed to be probability measures. Of course, these measures may have different masses inside each family. Then there appear naturally four different optimization problems (which are the same in the case of probability measures). For example, if the alternative contains measures with different masses, then maximizing the smallest "power" or minimizing the largest "probability of type II error" are different problems. A similar manipulation with a constraint on "the probability of type I error" provides two other problems. For all four problems, we find a dual optimization problem, whose solution gives a "least favorable" pair of measures determining a solution to the corresponding initial problem.

Though this four optimization problems are the same in the case of probability measures, the corresponding dual problems, being very similar, are different. Moreover, the existence of solutions of dual problems is proved under different assumptions. The dual optimization problem introduced in [6] is one of these four problems which requires the weakest assumptions.

The paper is organized as follows. Our main results are stated in Section 2. We deal with the case of arbitrary finite measures. Theorem 1 says that a minimax (or maximin) test always exists in all four problems. We also formulate dual minimization problems and show that there are no duality gaps. To ensure the existence of solutions to dual problems, in general, one should extend the domain of definition of dual problems. Conditions under which duality relations preserve and saddle points exist are given in Theorem 2. Section 3 contains the proofs. Intermediate statements are presented in a number of lemmas, which might be useful if the assumptions of Theorem 2 are not satisfied.

The families of measures corresponding to the null hypothesis and the alternative are assumed to be dominated. This allows us to identify a measure with its density and thus to consider the null hypothesis and the alternative as subsets $\mathcal{H}$ and $\mathcal{G}$ respectively of a positive cone $\mathbb{L}^1_+$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. E stands for expectation with respect to P, while expectation with respect to a measure Q is denoted by E_{Q}. Here and below $\mathbb{L}^1$ and $\mathbb{L}^\infty$ are the Banach spaces $\mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ of P-integrable and P-essentially bounded random variables respectively (random variables that coincide P-a.s. are identified) with the usual norms, whereas $\mathbb{L}^0 = \mathbb{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ is the space of real-valued random variables equipped with the topology of convergence in P-probability. Let us emphasize that infinite values are not allowed for elements of $\mathbb{L}^0$. Let Φ be the set of all randomized tests, that is, of measurable functions $\varphi\colon \Omega \to [0,1]$. Finally, denote by $\operatorname{co}(\cdot)$ the convex hull of a set of random variables. The bar over a subset of $\mathbb{L}^0$ stands for its closure.

Let us briefly describe how such optimization problems appear in mathematical finance. Assume that the discounted price process of d underlying assets is described as an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$. Let $\mathcal{P}_\sigma$ denote the set of probability measures P^* equivalent to P such that S is a σ-martingale with respect to P^*. A self-financing strategy is a pair (V_0, η) where $V_0 \geq 0$ is an initial capital and η is a predictable process such that the value process

$$V_t = V_0 + \int_0^t \eta_s \, dS_s, \quad t \in [0,T],$$

is well defined. Such a strategy is called admissible if the corresponding value process V satisfies $V_t \geq 0$ for all $t \in [0,T]$.

Consider a contingent claim with payoff H, where H is an $\mathcal{F}_T$-measurable nonnegative random variable. The superhedging price U_0 is defined as the smallest amount V_0 such that there exists an admissible strategy (V_0, η) with $V_T \geq H$. The corresponding strategy is called the superhedging strategy of the claim H. The dual characterization of the superhedging price U_0 is

$$U_0 = \sup_{\mathsf{P}^* \in \mathcal{P}_\sigma} \mathsf{E}_{\mathsf{P}^*} H < +\infty.$$

From a practical point of view the cost of superhedging is often too high, see e.g. [7]. For this reason, it is natural to study the possibility of investing less capital than the superhedging price. Namely, let $\tilde{V}_0 < U_0$ be a given maximal amount of money the investor is willing to spend. Then we look for an admissible strategy (V_0, η) with $0 < V_0 \leq \tilde{V}_0$ that minimizes the risk of losses due to the shortfall $\{V_T < H\}$. The size of a shortfall $(H - V_T)^+$ can be written as $(1-\varphi)H$, where

$$\varphi = \mathbb{1}_{\{H \leq V_T\}} + \frac{V_T}{H} \mathbb{1}_{\{H > V_T\}}$$

is called a success ratio. It is a randomized test, and in this way tests come into consideration. If φ is a non-randomized test, $\mathsf{E}\varphi$ is just the probability $\mathsf{P}(V_T \geq H)$ that the hedge is successful. Typically, the corresponding dynamic optimization problem can be split in two problems: to find a modified claim $\widetilde{\varphi}H$, where $\widetilde{\varphi}$ is a randomized test solving a corresponding static optimization problem and to find a superhedging strategy for the modified claim $\widetilde{\varphi}H$.

As an example, let us mention a static optimization problem appearing in [4]. It is required to maximize the expected success ratio $\mathsf{E}\varphi$ over tests $\varphi \in \Phi$ satisfying the constraint $\int \varphi H\, d\mathsf{P}^* \leq \tilde{V}_0$, where P^* runs over the class $\mathcal{P}_\sigma$. The constraint corresponds to the requirement $\mathsf{E}_{\mathsf{P}^*}[V_T] \leq \tilde{V}_0$. Here we have a composite null hypothesis $\mathcal{H} = \left\{H\frac{d\mathsf{P}^*}{d\mathsf{P}} : \mathsf{P}^* \in \mathcal{P}_\sigma\right\}$ and the simple alternative $\mathcal{G} = \{1\}$. This optimization problem corresponds both to Problem 1 and 2 introduced in the following section.

We also mention the papers [12], [13], and [17], where the shortfall risk is measured by a coherent risk measure. As a result, the aim is to minimize

$$\sup_{\mathsf{Q}\in\mathcal{Q}} \mathsf{E}_{\mathsf{Q}}[(H - V_T)^+] = \sup_{\mathsf{Q}\in\mathcal{Q}} \mathsf{E}_{\mathsf{Q}}[H(1-\varphi)],$$

where $\mathcal{Q}$ ia a set of probability measures. Here we obtain Problem 2 with the same composite null hypothesis $\mathcal{H}$ as above and a composite alternative $\mathcal{G} = \left\{H\frac{d\mathsf{Q}}{d\mathsf{P}} : \mathsf{Q} \in \mathcal{Q}\right\}$.

We also refer to [5, 8, 11, 14–16, 19, 20] for further related results on partial hedging, in particular, for other measures that quantify the shortfall risk.

2. Main results

Let α and β be real numbers. Put

$$\Phi_\alpha := \left\{\varphi \in \Phi \mid \sup_{h\in\mathcal{H}} \mathsf{E}[h\varphi] \leq \alpha\right\}, \quad \check{\Phi}_\beta := \left\{\varphi \in \Phi \mid \inf_{h\in\mathcal{H}} \mathsf{E}[h(1-\varphi)] \geq \beta\right\}.$$

The sets Φ_α and $\check{\Phi}_\beta$ are nonempty if $\alpha \geq 0$ and $\beta \leq \inf_{h\in\mathcal{H}} \mathsf{E}[h]$, respectively.

We consider the following optimization problems:

$$\text{to maximize} \quad \inf_{g\in\mathcal{G}} \mathsf{E}[g\varphi] \qquad \text{over} \quad \varphi \in \Phi_\alpha, \tag{1}$$

$$\text{to minimize} \quad \sup_{g\in\mathcal{G}} \mathsf{E}[g(1-\varphi)] \qquad \text{over} \quad \varphi \in \Phi_\alpha, \tag{2}$$

$$\text{to maximize} \quad \inf_{g\in\mathcal{G}} \mathsf{E}[g\varphi] \qquad \text{over} \quad \varphi \in \check{\Phi}_\beta, \tag{3}$$

$$\text{to minimize} \quad \sup_{g\in\mathcal{G}} \mathsf{E}[g(1-\varphi)] \qquad \text{over} \quad \varphi \in \check{\Phi}_\beta. \tag{4}$$

These problems will be referred to as Problems 1, 2, 3, and 4, respectively.

It is convenient to consider all the problems as maximization problems, so we change sign in Problems 2 and 4. Moreover, let us introduce the following unifying notation in the table below:

Table 1. The definition of α_i, G_i, H_i, F_i

	$i=1$	$i=2$	$i=3$	$i=4$
α_i	α		$-\beta$	
$G_i(g,\varphi)$	$\mathsf{E}[g\varphi]$	$\mathsf{E}[g(\varphi-1)]$	$\mathsf{E}[g\varphi]$	$\mathsf{E}[g(\varphi-1)]$
$H_i(h,\varphi)$	$\mathsf{E}[h\varphi]$	$\mathsf{E}[h\varphi]$	$\mathsf{E}[h(\varphi-1)]$	$\mathsf{E}[h(\varphi-1)]$
$F_i(g,h)$	$\mathsf{E}\big[(g-h)^+\big]$	$-\mathsf{E}\big[g\wedge h\big]$	$\mathsf{E}\big[g\vee h\big]$	$\mathsf{E}\big[(g-h)^-\big]$

The functions G_i and H_i are defined on $\mathbb{L}^1 \times \mathbb{L}^\infty$, and F_i are defined on $\mathbb{L}^1 \times \mathbb{L}^1$; note, however, that they are also well defined if g and h are nonnegative random variables and $\varphi \in \Phi$. Note also that differences between different F_i are linear functionals:

$$(x-y)^+ = -(x \wedge y) + x = x \vee y - y = (x-y)^- + (x-y).$$

We can now rewrite the problems (1)–(4) in the form

$$\text{to maximize} \quad \inf_{g \in \mathcal{G}} G_i(g,\varphi) \quad \text{over} \quad \varphi \in \Phi_i := \bigl\{\varphi \in \Phi \mid \sup_{h \in \mathcal{H}} H_i(h,\varphi) \le \alpha_i\bigr\}, \tag{5}$$

$i = 1,2,3,4$.

Theorem 1. *Let $\mathcal{G}$ and $\mathcal{H}$ be nonempty subsets of $\mathbb{L}^1_+$, $\alpha_i \ge 0$ in Problems 1 and 2 and $\alpha_i \ge -\inf_{h\in\mathcal{H}} \mathsf{E}[h]$ in Problems 3 and 4. Then, for every $i = 1,2,3,4$, there exists a randomized test $\widetilde{\varphi} \in \Phi_i$ which attains the supremum in*

$$\underline{v}_i := \sup_{\varphi \in \Phi_i} \Bigl(\inf_{g \in \mathcal{G}} G_i(g,\varphi)\Bigr). \tag{6}$$

Define also the dual minimization problem

$$\overline{v}_i := \inf_{(g,h)\in \mathrm{co}(\mathcal{G})\times \mathrm{co}(\mathcal{H}),\, z\ge 0} \Bigl(F_i(g,zh) + \alpha_i z\Bigr). \tag{7}$$

Then $\underline{v}_i = \overline{v}_i$.

It may happen that the minimization problem (7) has no solution. In Theorem 2 we establish sufficient assumptions for the (extended) minimization problem

$$\inf_{(g,h)\in \overline{\mathrm{co}(\mathcal{G})}\times \overline{\mathrm{co}(\mathcal{H})},\, z\ge 0} \Bigl(F_i(g,zh) + \alpha_i z\Bigr). \tag{8}$$

to have a solution with the same value.

(A): $\alpha_i > 0$ in Problems 1 and 2; $\alpha_i > -\inf_{h\in\mathcal{H}} \mathsf{E}[h]$ in Problems 3 and 4;
(H1): the family $\mathrm{co}(\mathcal{H})$ is bounded in P-probability;
(G1): the family $\mathcal{G}$ is bounded in $\mathbb{L}^1$;
(H2): the family $\mathcal{H}$ is uniformly integrable;
(G2): the family $\mathcal{G}$ is uniformly integrable;
(GH): the family $\bigl\{g \wedge h \colon g \in \mathrm{co}(\mathcal{G}),\, h \in \mathrm{co}(\mathcal{H})\bigr\}$ is uniformly integrable.

The uniform integrability of $\bigl\{g \wedge h \colon g \in \mathcal{G},\, h \in \mathcal{H}\bigr\}$ is not sufficient for (GH).

Theorem 2. *Let $\mathcal{G}$ and $\mathcal{H}$ be nonempty subsets of $\mathbb{L}^1_+$. Assume that there hold*

- (A), (H1), *and* (G2) *in Problem* 1;
- (A), (H1), (G1), *and* (GH) *in Problem* 2;
- (A), (G2), *and* (H2) *in Problem* 3;
- (A), (G1), *and* (H2) *in Problem* 4.

Then, for every $i = 1,2,3,4$, the infimum in (8) is attained and

$$\overline{v}_i = \inf_{(g,h)\in \overline{\mathrm{co}(\mathcal{G})}\times \overline{\mathrm{co}(\mathcal{H})},\, z\ge 0} \Bigl(F_i(g,zh) + \alpha_i z\Bigr) > -\infty. \tag{9}$$

For arbitrary $\varphi \in \Phi_i$ and $(g,h,z) \in \overline{\mathrm{co}(\mathcal{G})} \times \overline{\mathrm{co}(\mathcal{H})} \times \mathbb{R}_+$ the following are equivalent:

(i) *φ is a solution to (5) and (g,h,z) is a solution to (8).*

(ii)

$$\varphi = \begin{cases} 1, & \text{if } g > zh, \\ 0, & \text{if } g < zh, \end{cases} \qquad \mathsf{P}\text{-a.s.}, \tag{10}$$

$$H_i(h,\varphi) = \alpha_i \quad \text{if} \quad z > 0, \tag{11}$$

$$G_i(g,\varphi) \le G_i(g',\varphi) \quad \text{for all} \quad g' \in \mathcal{G}. \tag{12}$$

If $\mathsf{E}[h]$ is constant on $\mathcal{H}$ (in particular, if the family $\mathcal{H}$ consists of probability densities), then Problem 1 coincides with Problem 3 and Problem 2 coincides with Problem 4 provided $\alpha + \beta = \mathsf{E}[h]$. The dual problems (7) in the corresponding pairs also coincide. However, the dual problems (8) are different for different i in general. Thus we have a possibility to choose the problem inside the corresponding pair such that requires weaker assumptions in Theorem 2. Obviously, the assumptions corresponding to Problem 1 are weaker than those corresponding to Problem 3, and Problem 2 is preferable than Problem 4. Similarly, if $\mathsf{E}[g]$ is constant on $\mathcal{G}$ (in particular, if the family $\mathcal{G}$ consists of probability densities) then Problem 1 coincides with Problem 2 and Problem 3 coincides with Problem 4. Assumptions of Theorem 2 are weaker in Problems 2 and 4 than in Problems 1 and 3 respectively. Finally, if $\mathsf{E}[h]$ is constant on $\mathcal{H}$ and $\mathsf{E}[g]$ is constant on $\mathcal{G}$ (in particular, if the families $\mathcal{H}$ and $\mathcal{G}$ consist of probability densities), Problems 1–4 coincide, and the existence of a solution to the dual problem (8) is proved under weaker assumptions in Problem 2. This explains why the assumptions of Theorem 2 in the case of Problem 2 coincide with the assumptions of Theorem 1.1 (ii) in [6].

3. Proofs

Proof of Theorem 1. The proof is based on computations similar to those that were used in the proof of the corresponding statement in [6, p. 116]. We introduce the functions M_i and N on the product $X := \mathbb{L}^1 \times \mathbb{L}^1 \times \mathbb{R}$ of Banach spaces by

$$M_i(g,h,z) := F_i(g,h) + \alpha_i z, \qquad N(g,h,z) := \delta_{\mathrm{co}(\mathcal{G})}(g) + \delta_{\mathcal{J}}(h,z), \tag{13}$$

where $\mathcal{J} := \{(zh, z) \mid h \in \mathrm{co}(\mathcal{H}),\ z \in \mathbb{R}_+\}$ and δ is the convex indicator function: $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \notin A$. M_i and N are proper convex functions and M_i is finite and continuous everywhere. Hence, by the Fenchel–Rockafellar duality theorem [18],

$$\begin{aligned} \overline{v}_i &= \inf_{(g,h,z)\in X} \{M_i(g,h,z) + N(g,h,z)\} = \\ &= \max_{(\psi,\phi,x)\in X^*} \{-M_i^*(-\psi,-\phi,-x) - N^*(\psi,\phi,x)\}, \end{aligned} \tag{14}$$

where, as usual, "max" denotes a supremum which is attained, * means the Fenchel conjugate defined on the dual space $X^* := \mathbb{L}^\infty \times \mathbb{L}^\infty \times \mathbb{R}$, and the first equality is immediate from definitions. Let $(\widetilde{\psi}, \widetilde{\phi}, \widetilde{x})$ be a triple that attains the maximum in (14).

Calculating M_i^* and N^*, we get

$$\begin{aligned} M_i^*(\psi,\phi,x) :&= \sup_{(g,h,z)\in X} \big\{\mathsf{E}[g\psi + h\phi] + zx - M_i(g,h,z)\big\} = \\ &= \begin{cases} \delta_\Phi(\psi) + \delta_{\{-\psi\}}(\phi) + \delta_{\{\alpha_i\}}(x), & \text{if } i = 1, \\ \delta_{\Phi-1}(\psi) + \delta_{\{-\psi-1\}}(\phi) + \delta_{\{\alpha_i\}}(x), & \text{if } i = 2, \\ \delta_\Phi(\psi) + \delta_{\{1-\psi\}}(\varphi) + \delta_{\{\alpha_i\}}(x), & \text{if } i = 3, \\ \delta_{\Phi-1}(\psi) + \delta_{\{-\psi\}}(\varphi) + \delta_{\{\alpha_i\}}(x), & \text{if } i = 4, \end{cases} \end{aligned}$$

$$\begin{aligned} N^*(\psi,\phi,x) :&= \sup_{(g,h,z)\in X} \big\{\mathsf{E}[g\psi + h\phi] + zx - N(g,h,z)\big\} = \\ &= \sup_{g\in\mathrm{co}(\mathcal{G})} \mathsf{E}[g\psi] + \delta_{\{\sup_{h\in\mathrm{co}(\mathcal{H})} \mathsf{E}[h\phi] \le -x\}}(\phi,x) = \\ &= \sup_{g\in\mathcal{G}} \mathsf{E}[g\psi] + \delta_{\{\sup_{h\in\mathcal{H}} \mathsf{E}[h\phi] \le -x\}}(\phi,x). \end{aligned}$$

Thus, putting $\varphi := \phi + \mathbb{1}_{\{3,4\}}(i)$, we get

$$-M_i^*(-\psi, -\phi, -x) - N^*(\psi, \phi, x) = \\ = \begin{cases} \inf\limits_{g\in\mathcal{G}} G_i(g,\varphi), & \text{if } x = -\alpha_i,\ \psi = \mathbb{1}_{\{2,4\}}(i) - \varphi,\ \varphi \in \Phi_\alpha; \\ -\infty, & \text{otherwise.} \end{cases}$$

Hence, $\widetilde{\varphi} := \widetilde{\phi} + \mathbb{1}_{\{3,4\}}(i)$ is a solution to the maximization problem (5) and $\underline{v}_i = \overline{v}_i$. □

Everywhere below in this section the assumptions of Theorem 1 are supposed to be satisfied.

Lemma 1. *A test $\varphi \in \Phi_i$ is a solution to* (5) *if and only if there is a sequence $\{(g_n, h_n, z_n)\}$ in $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ such that*

$$F_i(g_n, z_n h_n) + \alpha_i z_n \to \inf_{g\in\mathcal{G}} G_i(g,\varphi). \tag{15}$$

There is a sequence $\{(g_n, h_n, z_n)\}$ in $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ satisfying (15) *and such that*

$$z_n \to z, \quad g_n \to g, \quad z_n h_n \to f \qquad \mathsf{P}\text{-}a.s., \tag{16}$$

where $z \in [0, +\infty]$, g and f are random variables with values in $[0, +\infty]$.

Proof. The first claim follows directly from Theorem 1. Take any sequence $\{(g_n', h_n', z_n')\}$ in $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ satisfying (15). Using the standard techniques of passing to forward-convex combinations based on [3, Lemma 9.8.1], one can construct a sequence $\{(g_n, f_n, z_n)\}$ such that $(g_n, f_n, z_n) \in \mathrm{co}(\{(g_n', z_n'h_n', z_n'), (g_{n+1}', z_{n+1}'h_{n+1}', z_{n+1}'), \dots\})$ for every n and

$$z_n \to z, \quad g_n \to g, \quad f_n \to f \qquad \mathsf{P}\text{-a.s.},$$

where z, g and f are as above. It is clear that $f_n = z_n h_n$ for some $h_n \in \mathrm{co}(\mathcal{H})$, while

$$\limsup_n \bigl\{F_i(g_n, f_n) + \alpha_i z_n\bigr\} \le \lim_n \bigl\{F_i(g_n', z_n'h_n') + \alpha_i z_n'\bigr\}$$

by the convexity of F_i. Hence, the sequence $\{(g_n, h_n, z_n)\}$ satisfies (15) as well by Theorem 1. □

Much of the subsequent arguments is based on the following two inequalities:

$$F_i(g, zh) + \alpha_i z \ge G_i(g,\varphi) + z\bigl\{\alpha_i - H_i(h,\varphi)\bigr\} \ge \inf_{g\in\mathcal{G}} G_i(g,\varphi). \tag{17}$$

The second one is true if, at least, $g \in \mathrm{co}(\mathcal{G})$, $h \in \mathrm{co}(\mathcal{H})$, $z \ge 0$, and $\varphi \in \Phi_i$. The first inequality is valid for integrable real variables g, h, real z, and $\varphi \in \Phi$ in view of the identities

$$F_i(g, zh) - \bigl\{G_i(g,\varphi) - zH_i(h,\varphi)\bigr\} = \mathsf{E}\bigl[(g - zh)^+(1-\varphi)\bigr] + \mathsf{E}\bigl[(g-zh)^-\varphi\bigr]. \tag{18}$$

Lemma 2. *Assume* (G1) *if $i = 2$ or $i = 4$. Then*

(i) $\underline{v}_i \in \mathbb{R}$;

(ii) *given $\varphi \in \Phi_i$ and a sequence $\{(g_n, h_n, z_n)\}$ in $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ satisfying* (16), *we have* (15) *if and only if*

$$\varphi = \begin{cases} 1, & \text{if } g > f, \\ 0, & \text{if } g < f, \end{cases} \qquad \mathsf{P}\text{-}a.s.,$$

the sequence $(g_n - z_n h_n)^+(1-\varphi) + (g_n - z_n h_n)^-\varphi$ is uniformly integrable,

$$z_n\bigl(\alpha_i - H_i(h_n,\varphi)\bigr) \to 0,$$
$$G_i(g_n,\varphi) \to \inf_{g\in\mathcal{G}} G_i(g,\varphi).$$

Proof. (i) is trivial, and (ii) follows from relations (17) and (18) applied to g_n, h_n, z_n, and φ after passing to the limit as $n \to \infty$. □

Lemma 3. *Assume* (G1) *if* $i = 2$ *or* $i = 4$, *and* (A). *If a sequence* $\{(g_n, h_n, z_n)\}$ *in* $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ *satisfies* (15) *and* (16), *then* $z < +\infty$.

Proof. The proof is immediate in all four cases. For example, in Problem 4

$$F_i(g_n, z_n h_n) + \alpha_i z_n \geq z_n \mathsf{E}[h_n] - \mathsf{E}[g_n] + \alpha_i z_n \geq z_n \Big(\inf_{h\in\mathcal{H}} \mathsf{E}[h] + \alpha_i\Big) - \sup_{g\in\mathcal{G}} \mathsf{E}[g],$$

which shows that z_n must be bounded to have a finite limit in (15). □

Lemma 4. *Assume* (H1). *If a sequence* $\{(g_n, h_n, z_n)\}$ *in* $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ *satisfies* (15) *and* (16), *and* $z < +\infty$, *then* $f = zh$, *where a random variable* h *is in* $\overline{\mathrm{co}(\mathcal{H})}$ *and takes finite values.*

Proof. Obviously, if $z > 0$, then $h = f/z = \lim_n h_n$ P-a.s., in particular, $h < +\infty$ P-a.s. in view of (H1). Let $z = 0$. Due to (H1), we have $f = 0$, and one may take any element in $\overline{\mathrm{co}(\mathcal{H})}$ as h. □

Proof of Theorem 2. Let $\varphi \in \Phi_i$ be a solution to (5), $g \in \overline{\mathrm{co}(\mathcal{G})}$, $h \in \overline{\mathrm{co}(\mathcal{H})}$, $z \in \mathbb{R}_+$. Then

$$G_i(g, \varphi) \geq \underline{v}_i \tag{19}$$

in view of (G2) if $i = 1$ or $i = 3$, and by Fatou's lemma if $i = 2$ or $i = 4$. Similarly,

$$H_i(h, \varphi) \leq \alpha_i \tag{20}$$

by Fatou's lemma if $i = 1$ or $i = 2$, and due to (H2) if $i = 3$ or $i = 4$. In particular, $G_i(g, \varphi) > -\infty$ and $H_i(h, \varphi) < +\infty$. Hence, the middle term in (17) is well defined and the second inequality in (17) is true. This also implies that $F_i(g, zh) > -\infty$ and the first inequality in (17) takes place. Thus, we have (9).

Next, using Lemma 1, take a sequence $\{(g_n, h_n, z_n)\}$ in $\mathrm{co}(\mathcal{G}) \times \mathrm{co}(\mathcal{H}) \times \mathbb{R}_+$ satisfying (15) and (16), and let h be as in Lemma 4. Then it follows from (15) that

$$F_i(g, zh) \leq \underline{v}_i. \tag{21}$$

Indeed, if $i \neq 2$, this is a consequence of Fatou's lemma, and (GH) is used if $i = 2$. Observe that, for $i = 1$ or $i = 3$, we have $g < +\infty$ P-a.s. because $F(g, zh) = +\infty$ otherwise. For $i = 2$ or $i = 4$, g is even integrable due to (G1). Hence $g \in \overline{\mathrm{co}(\mathcal{G})}$. Combining (17) and (21), we can conclude now that (g, h, z) attains the infimum in (8).

It remains to observe that (i) takes place if and only if

$$F_i(g, zh) + \alpha_i z = \inf_{g\in\mathcal{G}} G_i(g, \varphi).$$

In view of (17), (18), (19), and (20), the latter is equivalent to (ii). □

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References

1. V. Baumann, *Eine parameterfreie Theorie der ungünstigsten Verteilungen für das Testen von Hypothesen*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **11** (1968), no. 1, 41–60.
2. J. Cvitanić, I. Karatzas, *Generalized Neyman-Pearson lemma via convex duality*, Bernoulli, **7** (2001), no. 1, 79–97.
3. F. Delbaen, W. Schachermayer, *The Mathematics of Arbitrage*, Springer, Berlin, Heidelberg, 2006.
4. H. Föllmer, P. Leukert, *Quantile hedging*, Finance & Stochastics, **3** (1999), no. 3, 251–273.
5. H. Föllmer, P. Leukert, *Efficient hedging: Cost versus shortfall risk*, Finance & Stochastics, **4** (2000), no. 2, 117–146.
6. A. A. Gushchin, *A characterization of maximin tests for two composite hypotheses*, Mathematical Methods of Statistics, **24** (2015), no. 2, 110–121.

7. A. A. Gushchin, E. Mordecki, *Bounds on option prices for semimartingale market models*, Proceedings of the Steklov Institute of Mathematics, **237** (2002), 73–113.
8. D. Hernández-Hernández, E. Trevino-Aguilar, *Efficient hedging of European options with robust convex loss functionals: A dual-representation formula*, Mathematical Finance, **21** (2011), no. 1, 99–115.
9. O. Krafft, H. Witting, *Optimale Tests und ungünstigste Verteilungen*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **7** (1967), no. 4, 289–302.
10. E. L. Lehmann, *Testing Statistical Hypotheses*, Wiley, New York, 1959.
11. A. Melnikov, A. Nosrati, *Equity-Linked Life Insurance: Partial Hedging Methods*, Chapman & Hall/CRC, Boca Raton, 2017.
12. Y. Nakano, *Minimizing coherent risk measures of shortfall in discrete-time models under cone constraints*, Applied Mathematical Finance, **10** (2003), no. 2, 163–181.
13. Y. Nakano, *Efficient hedging with coherent risk measure*, Journal of Mathematical Analysis and Applications **293** (2004), no. 1, 345–354.
14. Y. Nakano, *Partial hedging for defaultable claims*, Advances in Mathematical Economics, **14** (2011), 127–145.
15. A. A. Novikov, *Hedging of options with a given probability*, Theory of Probability and Its Applications, **43** (1999), no. 1, 135–143.
16. B. Rudloff, *Convex hedging in incomplete markets*, Applied Mathematical Finance, **14** (2007), no. 5, 437–452.
17. B. Rudloff, *Coherent hedging in incomplete markets*, Quantitative Finance, **9** (2009), no. 2, 197–206.
18. R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Mathematical Journal, **33** (1966), no. 1, 81–89.
19. E. Treviño Aguilar, *Robust efficient hedging for American options: The existence of worst case probability measures*, Statistics & Decisions, **27** (2009), no. 1, 1–23.
20. E. Treviño Aguilar, *Duality in a problem of static partial hedging under convex constraints*, SIAM Journal on Financial Mathematics, **6** (2015), 1152–1170.

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ПЕРЕВІРКА ГІПОТЕЗ ДЛЯ МІР ІЗ РІЗНИМИ МАСАМИ: ЧОТИРИ ОПТИМІЗАЦІЙНІ ЗАДАЧІ

О. О. ГУЩІН, С. С. ЛЕЩЕНКО

Анотація. Ми розглядаємо задачу, аналогічну перевірці двох складних гіпотез, де міри, які складають гіпотези, не є ймовірнісними та можуть мати різні маси. Тоді природно розглянути чотири різні оптимізаційні задачі. Для характеризації оптимальних розв'язків ми вводимо відповідні двоїсті оптимізаційні задачі. Наша головна мета — знайти достатні умови існування сідлових точок у кожній задачі.