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Density estimation by observations with admixture

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Abstract

A semiparametric two-component mixture model is considered, in which the distribution of one (primary) component is unknown and assumed symmetric. The distribution of the other component (admixture) is known. Kernel type estimates for the density of the primary component are considered. Asymptotic normality of the estimates is demonstrated.

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1. Introduction

In this paper we consider i.i.d. real valued observations $\{\xi_1, \xi_2, ..., \xi_N\}$ taken from a mixture of two components: the primary component with an unknown distribution and the admixture whose distribution is known. It will be assumed that both components have probability density function (pdf) and the pdf of the primary component is symmetric around it's median. Thus the pdf of ξ_i is

$$\psi(x) = pf(x-a) + (1-p)f_0(x), \tag{1}$$

where $f_0(x)$ is the known admixture pdf, $a \in R$ is the median of the primary component, $f(x)$ is the pdf of the primary component centered by a (f is an even function: $f(-x) = f(x)$) and $p \in (0,1)$ is the mixing probability. The parameters a and p (which are called Euclidean parameters) are assumed to be unknown. The pdf f is the nonparametric part of the model. There is a vast literature devoted to the estimation of parameters and distributions of two-component mixture [1-3].

The model (1) was introduced in [1] where estimates for a and p are constructed and their consistency is demonstrated. Another estimate for a based on generalized estimating equations (GEE) approach was proposed in [3]. Here asymptotic normality of this estimate is demonstrated.

An estimate $\hat{f}$ for f is proposed in [1] which utilizes the estimates for a and p obtained in this article and a kernel estimate for ψ. Consistency of this estimate in L_1- norm is demonstrated.

The aim of this paper is to investigate asymptotic behavior of $\hat{f}$ and it's modification in the case when the true pdf f is twice continuously differentiable.

2. Problem setting

To derive an estimate for f note that by (1)

$$f(x) = \frac{1}{p}\left(\psi(x+a) - (1-p)f_0(x+a)\right). \tag{2}$$

We will replace the unknown parameters at this formula by their estimates. For the estimation of pdf $\psi(x)$ we use the kernel estimate [6,p.57]

$$\hat{\psi}_N(x) = \frac{1}{Nh_N}\sum_{j=1}^{N} K\left(\frac{x-\xi_j}{h_N}\right), \tag{3}$$

where K is a kernel, that is pdf on R; $\{h_N, N \geq 1\}$ is a bandwidth such that $h_N \to 0$ and $Nh_N \to \infty$ as $N \to \infty$.

An estimate for a was proposed in paper [3]: $\hat{a}_N$ is a root to the equation

$$\hat{h}(\hat{a}_N) = 0, \tag{4}$$

where
$$\hat{h}(\alpha) = \frac{1}{N}\sum_{j=1}^{N}\left(g_1(x-\alpha)G_2(\alpha) - g_2(x-\alpha)G_1(\alpha)\right);$$
$g_1(x)$, $g_2(x)$ are some fixed odd functions; $G_i(\alpha) = \int_{-\infty}^{\infty} g_i(x-\alpha)f_0(x)dx$.

To estimate the parameter p we use the method of moments. Let us denote $m_0 = \int_{-\infty}^{\infty} x f_0(x)dx$. Equating the theoretical and empirical moments of observations, we obtain the equation for the estimate $\hat{p}_N$:
$$\hat{p}(a - m_0) + m_0 = \frac{1}{N}\sum_{j=1}^{N}\xi_j.$$

Since a is unknown, we replace it by a GEE estimate $\hat{a}_N$ from (4). So,

(5)
$$\hat{p}_N = \frac{1}{\hat{a}_N - m_0}\left(\frac{1}{N}\sum_{j=1}^{N}\xi_j - m_0\right).$$

The resulting estimate for f is

(6)
$$\hat{f}_N(x) = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x + \hat{a}_N) - (1 - \hat{p}_N)f_0(x + \hat{a}_N)\right).$$

This estimate has the same form as the estimate proposed in [1], but we use here another estimates for a and p. Additionally we consider a symmetrized version of our estimate:

(7)
$$\tilde{f}_N(x) = \frac{\hat{f}_N(x) + \hat{f}_N(-x)}{2}.$$

3. Main results

Different numerical constants will be denoted by $c_1, c_2, ..., c_k$. The sign $\Rightarrow$ means the weak convergence of distributions.

To derive asymptotic results we will use the following assumptions.

(i) Assumptions on the kernel:

$K(x)-$ is a finitely supported function, i.e. $\exists\,[a,b] \in R$, that $K(x) = 0$ if $x \notin [a,b]$;
$$Var_R K'(x) < c_1, \text{where Var means the functional variation on } [a,b];$$
$$D = \int_{-\infty}^{\infty} z^2 K(z)dz < \infty;\quad d^2 = \int_{-\infty}^{\infty} K^2(z)dz < \infty;\quad \int_{-\infty}^{\infty} zK(z)dz = 0.$$

(ii) Assumptions on density's components:

$f(x)$, $f_0(x)$ are twice continuously differentiable;
$$|f'(x)| < c_2;\qquad |f_0'(x)| < c_3;\qquad m_0 \neq a;$$
$$\int_{-\infty}^{\infty} x^2 f(x)dx < \infty;\qquad \int_{-\infty}^{\infty} x^2 f_0(x)dx < \infty.$$

(iii) Assumptions on the estimating functions for a:

$g_i'(x),\ G_i'(x)$ are continuous on R;

$$E(g_i(\xi_1 - a))^2 < \infty, i = 1,2; \qquad E\frac{\partial}{\partial\alpha}h(\xi_1,\alpha)|_{\alpha=a} \neq 0;$$

for some $\varepsilon > 0, \delta > 0$:

$$E \sup_{\alpha:|a-\alpha|<\varepsilon} (g_i'(\xi_1 - \alpha))^{1+\delta} < \infty, i = 1,2.$$

(iv) $\hat{a}_N$ is consistent estimate for a;

(iiv) Assumption on bandwidth h_N:

$$h_N = CN^{-1/5},\ C - \textit{some constant}.$$

Theorem 1. *Assume that the assumptions (i)-(iiv) hold.*

Then

$$N^{\frac{2}{5}}\left(\hat{f}_N(x) - f(x)\right) \Rightarrow \eta_1,$$

where η_1 *- is a normal random variable, distributed as*

$$N\left(\frac{D^2C^2}{2p}\psi''(x+a), \frac{d^2}{Cp^2}\psi(x+a)\right).$$

The next theorem describes the asymptotic behavior of the symmetrized estimate (7).

Theorem 2. *Assume that (i)-(iiv) hold. Then*

$$N^{\frac{2}{5}}\left(\widetilde{f}_N(x) - f(x)\right) \Rightarrow \eta_2,$$

where η_2 *- normal random variable, distributed as*

$$N\left(\frac{D^2C^2}{4p}\left(\psi''(x+a) + \psi''(-x+a)\right), \frac{d^2}{4Cp^2}\left(\psi(x+a) + \psi(-x+a)\right)\right).$$

4. Proofs of results

Proof. of theorem 1.

From

$$f(x) = f_0(x+a) + \frac{1}{p}(\psi(x+a) - f_0(x+a)),$$

and (5) we get

$$\hat{f}_N(x) - f(x) = \varepsilon_1^N + \varepsilon_2^N + \varepsilon_3^N + \varepsilon_4^N,$$

where

$$\varepsilon_1^N = \left(1 - \frac{1}{\hat{p}_N}\right)(f_0(x+\hat{a}_N) - f_0(x+a));$$

$$\varepsilon_2^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+\hat{a}_N) - \psi(x+\hat{a}_N)\right); \quad \varepsilon_3^N = \frac{1}{\hat{p}_N}\left(\psi(x+\hat{a}_N) - \psi(x+a)\right);$$

$$\varepsilon_4^N = \frac{p - \hat{p}_N}{p\hat{p}_N}(\psi(x+a) - f_0(x+a)). \tag{8}$$

At first we will consider convergence rate of Euclidean parameters estimates $\hat{a}_N$ and $\hat{p}_N$. By the theorem 2.1 from [3]: if (iii) holds

$$\hat{a}_N - a = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{9}$$

Remark 1. For any random sequence ε_N the notation $\varepsilon_N = O_p\left(\frac{1}{\sqrt{N}}\right)$ means, that

$$\lim_{C\to\infty} \limsup_{N\to\infty} \Pr\left\{\sqrt{N}|\varepsilon_N| > C\right\} = 0.$$

Now we will consider estimate of parameter of p. From (5)

$$\hat{p}_N - p = \frac{1}{\hat{a}_N - m_0}\left(\frac{1}{N}\sum_{j=1}^{N}\xi_j - m_0(1-p) - ap\right) - \frac{p}{\hat{a}_N - m_0}\left(\hat{a}_N - a\right). \tag{10}$$

By the central limit theorem [4 , p.179]

$$\frac{1}{N}\sum_{j=1}^{N}\xi_j - m_0(1-p) - ap = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{11}$$

Equations (9)-(11) imply that

$$\hat{p}_N - p = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{12}$$

Due to (9), boundedness of derivatives of $\psi(x)$, $f_0(x)$ and Lagrangian theorem, we conclude that for some intermediate points between $\hat{a}_N$ and a

$$\varepsilon_1^N = \left(1 - \frac{1}{\hat{p}_N}\right) f_0'(x+\theta_1)(\hat{a}_N - a) = O_p\left(\frac{1}{\sqrt{N}}\right),$$

$$\varepsilon_3^N = \frac{1}{\hat{p}_N}\psi'(x+\theta_2)(\hat{a}_N - a) = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{13}$$

From (12) we derive the same convergence rate for ε_4^N.

$$\varepsilon_4^N = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{14}$$

It remains to consider ε_2^N. Let us expand it as

$$\varepsilon_2^N = \delta_1^N + \delta_2^N + \delta_3^N; \quad \text{where} \tag{15}$$

$$\delta_1^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+\hat{a}_N) - \hat{\psi}_N(x+a)\right),$$

(16)

$$\delta_2^N = -\frac{1}{\hat{p}_N}\left(\psi(x+\hat{a}_N) - \psi(x+a)\right),\ \delta_3^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+a) - \psi(x+a)\right).$$

Obviously, δ_2^N is $O_p\left(\frac{1}{\sqrt{N}}\right)$ as $N \to \infty$. Asymptotic behavior of third term is described by the next theorem.

Theorem 3. *[4, p.57] Assume, that*

$$D = \int_{-\infty}^{\infty} z^2 K(z)dz < \infty; \quad d^2 = \int_{-\infty}^{\infty} K^2(z)dz < \infty; \quad \int_{-\infty}^{\infty} zK(z)dz = 0;$$

function $\psi(x)$ is doubly differentiable. The estimate $\hat{\psi}(x)$ defined by (3) can be represented as

$$\hat{\psi}_N(x) = \psi_N(x) + \frac{\zeta_N(x)}{\sqrt{Nh_N}},$$

where

$$\psi_N(x) = E\hat{\psi}_N(x) = \int_{-\infty}^{\infty} K(z)\psi(x - zh_N)dz \to_{N\to\infty} \psi(x) \ ;$$

$$\zeta_N(x) = \frac{1}{\sqrt{Nh_N}} \sum_{j=1}^{N} \left(K\left(\frac{x-\xi_j}{h_N}\right) - EK\left(\frac{x-\xi_j}{h_N}\right)\right) \Rightarrow \zeta;$$

where ζ is a zero mean Gaussian r.v. with variance $d^2\psi(x)$.

By Tailor's formula

$$\psi_N(x) = \int_{-\infty}^{\infty} K(z)\psi(x - zh_N)dz =$$

$$= \int_{-\infty}^{\infty} K(z)\left(\psi(x) - zh_N\psi'(x) + \frac{z^2{h_N}^2}{2}\psi''(\theta_{x,h_N})\right)dz =$$

$$= \psi(x) + \frac{{h_N}^2}{2}\int_{-\infty}^{\infty} z^2K(z)\psi''(\theta_{x,h_N})dz = \psi(x) + \frac{{h_N}^2D^2}{2}\psi''(x) + o({h_N}^2).$$

So, by theorem 3

$$\hat{\psi}_N(x+a) - \psi(x+a) = \frac{D^2\psi''(x+a)}{2}h_N^2 + \frac{\zeta_N(x+a)}{\sqrt{Nh_N}} + o(h_N^2),$$

where $\zeta_N(x+a)$ converges weakly to $\zeta(x+a)$, distributed as $N(0, d^2\psi(x+a))$. By assumption (iiv) $h_N = CN^{-1/5}$, so

$$\delta_3^N = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) +$$

$$+\frac{N^{-2/5}}{p\hat{p}_N}(p - \hat{p}_N)\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o(N^{-2/5}) =$$

$$= \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o(N^{-2/5}). \tag{17}$$

For the first term in (15) we have

$$\delta_1^N = \frac{\hat{\psi}'_N(x+\theta_3)}{\hat{p}_N}(\hat{a}_N - a), \tag{18}$$

where θ_3 is an intermediate point between $\hat{a}_N$ and a.

Consider

$$\hat{\psi}'_N(x) = \frac{1}{Nh_N^2}\sum_{j=1}^{N} K'\left(\frac{x-\xi_j}{h_N}\right) = \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) d\mu_N(y),$$

where $\mu_N(y) = \frac{1}{N}\sum_{j=1}^{N} I\{\xi_j < y\}$ – is the empirical measure ($I\{A\}$ is an indicator of a set A). Similarly denote

$$\tilde{\psi}_N(x) = \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) dP(y),$$

where $P(x) = \Pr\{\xi_1 < x\}$. Then

$$\hat{\psi}'_N(x) = \tilde{\psi}_N(x) + \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) d\left(\mu_N(y) - P(y)\right). \tag{19}$$

Changing variables and integrating by parts we obtain

$$\tilde{\psi}_N(x) = \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right)\psi(y)dy = \int_{-\infty}^{\infty} K(z)\psi'(x-zh_N)dz = \psi'(x)+O(h_N^2).$$

To estimate δ_1^N we need the following inequality.

Lemma 1. *([5, p.138]). If $f(x)$ is a continuous on $[a,b]$ function and $g(x)$ is a function with bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(x)dg(x)\right| \le \max_{x\in[a,b]} |f(x)| \cdot Var_{x\in[a,b]} g(x).$$

So,

$$\frac{1}{h_N^2}\left|\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) d\left(\mu_N(y) - P(y)\right)\right| =$$

$$\frac{1}{h_N^2}\left|\int_{-\infty}^{\infty} \left(\mu_N(y) - P(y)\right) dK'\left(\frac{x-y}{h_N}\right)\right| \le$$

$$\le \frac{1}{h_N^2} var_{y\in R} K'\left(\frac{x-y}{h_N}\right) \cdot \sup_{y\in R}\left|\mu_N(y) - P(y)\right|.$$

By the assumption (i) $Var_{y\in R}K'\left(\frac{x-y}{h_N}\right)$ is finite. By the Vapvik-Cervonenkis inequality [6, p.231]

$$\Pr\left\{\sup_{y\in R}\left|\mu_N(y) - P(y)\right| > \varepsilon\right\} \le 6(2N+1)\exp\left(-\frac{\varepsilon^2(N-1)}{2}\right). \tag{20}$$

For $\varepsilon \ge \frac{\ln N}{\sqrt{N}}$, the right hand side of (20) tends to zero as $N \to \infty$. So we conclude, that

$$\left|\mu_N(y) - P(y)\right| = O_p\left(\frac{\ln N}{\sqrt{N}}\right).$$

Substituting derived results about asymptotics of both summands and $h_N = CN^{-1/5}$ to (19), we conclude, that

$$(21)\quad \hat{\psi}'_N(x) = \psi'(x) + o(h_N^2) + O_p\left(\frac{1}{h_N^2}\frac{\log N}{\sqrt{N}}\right) = \psi'(x) + O_p(\ln N \cdot N^{-1/10}).$$

This implies $\delta_1^N = O_p\left(\frac{1}{\sqrt{N}}\right)$. So,

$$(22)\qquad \varepsilon_2^N = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5})$$

Substituting (13), (14), (21) into (7), we derive statement of theorem 1 □

Proof. of theorem 2. Obviously,

$$(23)\qquad \tilde{f}_N(x) - f(x) = \frac{\hat{f}_N(x) - f(x)}{2} + \frac{\hat{f}_N(-x) - f(-x)}{2}.$$

Let us apply theorem 3 to the first and second terms of (23).

$$\hat{f}_N(x) - f(x) = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5});$$

(24)

$$\hat{f}_N(-x) - f(-x) = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(-x+a)}{2} + \frac{\zeta_N(-x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5}),$$

where $\zeta_N(x+a)$, $\zeta_N(-x+a)$ defined by:

$$\zeta_N(x+a) = \frac{1}{\sqrt{Nh_N}}\sum_{j=1}^{N}\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right);$$

$$\zeta_N(-x+a) = \frac{1}{\sqrt{Nh_N}}\sum_{j=1}^{N}\left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right).$$

Lemma 2. *If (i)-(vii) hold and $x \neq a$*

$$\begin{pmatrix}\zeta_N(x+a)\\ \zeta_N(-x+a)\end{pmatrix} \Rightarrow \begin{pmatrix}\zeta^1\\ \zeta^2\end{pmatrix}$$

where $\begin{pmatrix}\zeta^1\\ \zeta^2\end{pmatrix}$ *is a zero mean normal vector with covariance matrix*

$$B = d^2\begin{pmatrix}\psi(x+a) & 0\\ 0 & \psi(-x+a)\end{pmatrix}$$

Proof. of lemma 2. We will use the central limit theorem in the following form.

Theorem 4. *[7 , p. 201] Let $\xi_{1,N}, ..., \xi_{N,N}$ be random vectors independent for fixed N with $E\xi_{j,N} = 0$; $\zeta_N = \sum_{j=1}^N \xi_{j,N}$. Denote*

$$\sigma^2_{j,N} = E\xi_{j,N}\xi^T_{j,N}; \quad \sigma^2_N = \sum_{j=1}^N \sigma^2_{j,N}.$$

Assume that

1)$\sigma^2_N \to \sigma^2$, where σ^2 - some positive-definite matrix;

2) The Lindeberg condition holds:

$$B_N = \sum_{j=1}^N E\left(\xi^2_{j,N};\ |\xi_{j,N}| > \tau\right) \to 0;\ N \to \infty$$

for any constant $\tau > 0$. Then random vector ζ_N converges weakly to a zero mean normal vector with the covariance matrix σ^2.

Let us show that the assumptions of the theorem 4 hold for the vector sequence $\begin{pmatrix} \zeta_N(x+a) \\ \zeta_N(-x+a) \end{pmatrix}$. By theorem 1

$$D\zeta_N(x+a) \to d^2\psi(x+a); \quad D\zeta_N(-x+a) \to d^2\psi(-x+a).$$

For the covariance of the entries we have

$$Cov(\zeta_N(x+a), \zeta_N(-x+a)) =$$

$$= E\left(\sum_{j=1}^N \frac{1}{\sqrt{Nh_N}}\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)\times\right.$$

$$\left.\times\sum_{j=1}^N \frac{1}{\sqrt{Nh_N}}\left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)\right) =$$

$$= \frac{1}{Nh_N}\sum_{j=1}^N E\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)\left(K\left(\frac{-x+a-\xi_j}{h_N}\right) -\right.$$

$$\left.-EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right) = \frac{1}{h_N}\left(EK\left(\frac{x+a-\xi_1}{h_N}\right)K\left(\frac{-x+a-\xi_1}{h_N}\right) -\right.$$

$$\left.-EK\left(\frac{x+a-\xi_1}{h_N}\right)EK\left(\frac{-x+a-\xi_1}{h_N}\right)\right) = \int_{-\infty}^{\infty} K(s)K\left(s - \frac{2x}{h_N}\right)\times$$

$$\times\psi(x+a-sh_N)ds - h_N\int_{-\infty}^{\infty} K(s)\psi(x+a-sh_N)ds\int_{-\infty}^{\infty} K(s)\psi(-x+a-sh_N)ds.$$

Since $K(x)$ is finitely supported, $\psi(x)$ is bounded, $x \neq a$ and $h_N \to 0$,

$$Cov(\zeta_N(x+a), \zeta_N(-x+a)) \to 0,\ N \to \infty.$$

Then, by (i) $K(x) < c_4$. So by Tchebyshev inequality

$$B_N = \frac{1}{Nh_N}\sum_{j=1}^{N} E\left(\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$

$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2\right) \times$$

$$\times\chi\left\{\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$

$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2 > \tau^2 N h_N\right\} \le$$

$$\le \frac{8c_4^2}{h_N}\Pr\left\{\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$

$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2 > \tau^2 N h_N\right\} \le \frac{8c_4^2}{h_N}\frac{8c_4^2}{\tau^2 N h_N}$$

Since $h_N = CN^{-1/5}$, the rhs of inequality tends to zero. Lindeberg's condition is verified, lemma 2 proved. □

To complete the proof of theorem 2 it is sufficient to substitute (24) to (23) and take into account asymptotic of normal components. □

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